Problem assignment 5.

Introduction to Differential Geometry.

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[P] 1. Let M be a manifold and $\delta : \Omega(M) \to \Omega(M)$ an operator which satisfies the usual properties (linear of degree 1, satisfies super Leibnitz rule, $\delta^2 = 0$ and $\delta : \Omega^0 \to \Omega^1$ coincides with d).

Show that δ is equal to the DeRham differential d.

(**Hint.** Using cut-off functions show that δ induces similar operator on any open subset $U \subset M$).

2. Let $T : V \to V$ be a linear endomorphism of an n-dimensional vector space V. It induces an operator $T^* : Alt(V) \to Alt(V)$.

Show that on the space $Alt^n(V)$ this operator is multiplication by $\det(T)$.

3. Check that the multiplication in the algebra $Alt(V)$ is associative and super commutative.

4. Check that the DeRham differential d given by $d(f dx_{i_1}...dx_{i_k}) = df dx_{i_1}...dx_{i_k}$ satisfies super Leibnitz rule.

Definition. Let R be a rectangular and f a function on R. We say that f is a **step function** if there exists a partition P of R such that the function f takes constant value f_α in the interior of every part P_α of the partition P.

[P] 5. (i) Show that every step function f is integrable and its integral $I(f)$ equals $I(f) = \sum_\alpha f_\alpha vol(P_\alpha)$. Show that for any step function Fubini theorem holds.

(ii) Show that any integrable function can be approximated by step functions.

More precisely, show that a function f on the rectangle R is integrable iff for any $\varepsilon > 0$ there exist step functions f^- and f^+ such that $f^- \le f \le f^+$ and $I(f^+) \le I(f^-) + \varepsilon$.

(iii) Using result of (ii) prove Fubini's theorem for an integrable function $f(x, y)$ in two sets of variables.

Namely, for any bounded function h on a rectangle define lower and upper integrals $\int^- hD(x) = \sup L(P, h)$ and $\int^+ hD(x) = \inf U(P, h)$, where P runs over all partitions of a rectangle R.

Show that if $f(x, y)$ is an integrable function in two sets of variables x and y then the functions $g^-(y) = \int^- f(x, y)D(x)$ and $g^+(y) = \int^+ f(x, y)D(x)$ are both integrable and their integral equals to $\int f(x, y)D(x, y)$.

[P] 6. Show that any integrable function can be approximated by smooth functions. More precisely, suppose f is a function with compact support on $\mathbf{R}^n$.

Show that f is integrable iff for every $\varepsilon > 0$ there exist two smooth functions with compact support f^- and f^+ such that $f^- \le f \le f^+$ and $\int f^+ D(x) \le \int f^- D(x) + \varepsilon$.

Use this to show that the change of variables formula for the integration holds for any integrable function f.

[P] 7. Compute the integral $\int_T \omega$, where ω is a 2-form in $\mathbf{R}^3$ given by $\omega = dxdy + zdxdz$ and T is a closed 2-torus in $\mathbf{R}^3$. Can you propose a method how to compute the corresponding integral in the case when T is a surface with boundary in $\mathbf{R}^3$?

[P] 8. Let M, N be two manifolds of dimensions m and n. Suppose we fixed orientations μ and ν of M and N.

Show that this defines an orientation of the manifold $M \times N$ (it is called the product orientation).

Describe how the product orientations are compatible with the natural diffeomorphism $M \times N \to N \times M$.