Problem assignment 6.

Introduction to Differential Geometry.

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1. Let U be a domain and M a closed submanifold of U. Show that the restriction morphism $r : \Omega(U) \to \Omega(M)$ is onto.

2. Let M be a manifold of dimension n with orientation μ. Show that there exists a differential n-form ω on M such that at every point of M ω is not 0 and is compatible with μ.

3. Calculate the exterior derivative of the following forms in $\mathbf{R}^3$:
$z^2 \cdot dxdy + \left(z^2 + xy\right) dxdz$
$13xdx + y^3dy + xz \cdot dz$
$f \cdot dg$ where f, g are functions
$(x + 2y^3)(dzdx + 1/2dydx)$

4. Consider differential 1-form $\alpha = dz + xdy$ on $\mathbf{R}^3$
a) Compute $\eta = d\alpha$ and $\omega = \alpha d\alpha$.
b) Show that for any 2-dimensional submanifold $Y \subset \mathbf{R}^3$ the restriction of α to Y is not identically 0.

5. Let $D : \Omega(X) \to \Omega(X)$ be a linear operator of degree d (i.e. $D(\Omega^i) \subset \Omega^{i+d}$).
We say that D is **derivation** if it satisfies the Leibniz formula $D(\omega\alpha) = D\omega\alpha + (-1)^{d\cdot\deg(\omega)}\omega D\alpha$.
a)Show that if D is a derivation, $\omega \in \Omega$, then operator ωD is also a derivation.
b)Show, that if D and F are derivations of degrees d and f, then $[D, F] = DF - (-1)^{df} FD$ is a derivation of degree $d + f$.

6. Let ξ, η be vector fields on a manifold X, $\tau = [\xi, \eta]$ their commutator. Consider operator of interior multiplication $i_\xi : \Omega(X) \to \Omega(X)$ and Lie derivative $L_\xi = [d, i_\xi]$ (and similarly for η).
Compute $[L_\xi, i_\eta]$, $[L_\xi, L_\eta]$, $[i_\xi, i_\eta]$, $[d, i_\xi]$, $[d, L_\xi]$.

7. Let W be the algebra of differential forms with polynomial coefficients on $\mathbf{R}^n$. Describe all derivations of this algebra.

8. a) Let V be an n-dimensional space, $f \in V^*$, $f \neq 0$, and $H \subset V$ the corresponding hyperspace. Show that wedge product with f gives an isomorphism of $Alt^{n-1}(H^)$ and $Alt^n(V)$.
b) Let f be a smooth function on $\mathbf{R}^n$, X the set of its zeros. We assume that df does not vanish at points of X. Using a) show, that the volume form $\omega = dx_1 \dots dx_n$ on $\mathbf{R}^n$ defines canonical $(n-1)$-form η on X.
c) In case $n = 3$, X unit sphere, write explicitly the form η in some local coordinate system on X.

9. Consider the hypersurface $M \subset \mathbf{R}^3$ given by equation $f = 0$ where $f = x^2 + y^2 - z^2 + 1$.
(i) Compute the form $\eta = dxdydz/df$ on M in some coordinate system.
(ii) Show the relation of the volume form η to the area form on M induced by standard Euclidean metric on $\mathbf{R}^3$.

10. Consider differential $(n-1)$-form $\omega = x_1 dx_2 y dx_3 y \dots y dx_n$on $\mathbf{R}^n$. Show that $\int_{S^n} \omega \neq 0$. How to compute its value?