Problem assignment 1

Analysis on Manifolds

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Let M denote a manifold of dimension m (for example $\mathbf{R}^m$).

1. Let $f_1, ..., f_k$ be a collection of smooth functions on M. Denote by $Z = Z(f_1, ...f_k)$ the set of common zeroes of functions f_i.

Suppose that at some point $a \in Z$ differentials df_i of functions f_i are linearly independent. Show that then Z near the point a is a manifold. Describe the tangent space $T_a(Z)$.

2. Cut off functions. Show that for every $\delta > 0$ there exists a smooth function p on $\mathbf{R}^n$ such that

$0 \leq p \leq 1$, $p \equiv 0$ outside the ball B_1 of radius 1 around 0 and $p \equiv 1$ inside the ball $B_{1-\delta}$ of radius $1 - \delta$ around 0.

(**Hints**. First check that the function $u(t)$ in one variable given by $u(t) = 0$ for $t \leq 0$ and $u(t) = exp(-1/t^2)$ for $t > 0$ is smooth.

Then construct a monotone smooth function $h(t)$ in one variable such that $h(t) \equiv 0$ for $t \leq -\delta$ and $h(t) \equiv 1$ for $t > 0$.)

3. Hadamard's lemma. Consider the Euclidean space $V \cong \mathbf{R}^n$ with coordinates x_i.

Show that any smooth function $f \in C^\infty(V)$ can be written in the form $f = f(0) + \sum x_i h_i$, where h_i are smooth functions on V.

Hint. In case of one variable x we can define $h(x) = \int_0^1 u(tx)dt$, where $u = \frac{df}{dx}$.

4. Suppose that a morphism $\phi : X \to Y$ is transversal to a submanifold $Z \subset Y$. Then $W = \phi^{-1}(Z)$ is a submanifold of X. Consider also another morphism $\phi' : X \to Y'$ and assume that it is transversal to a submanifold $Z' \subset Y'$, so that $W' = \phi'^{-1}(Z')$ is a submanifold of X.

Show that the restriction of the morphism ϕ to $W' \subset X$ is transversal to Z iff the restriction of the morphism ϕ' to $W \subset X$ is transversal to Z'.

5. Let M be a manifold of dimension m and $\phi : K \to M$ a morphism of manifolds with $dim K < m$.

Show directly that the image $\phi(K) \subset M$ has measure 0.

(**Remark.** The example of Peano curve shows that this is not true for **continuous** maps.)

6. Construct an example of a smooth morphism $f : \mathbf{R} \to \mathbf{R}$ for which the set of the critical values is dense.

7. Using Sard's lemma show that for a manifold K of dimension $< n$ any morphism $\phi : K \to S^n$ can be contracted to a point.

(We will soon see that this implies that S^n is not diffeomorphic to $S^k \times S^{n-k}$ for $0 < k < n$.)