Problem assignment 2

Analysis on Manifolds

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1. (i) Let $Z \subset X \subset Y$ be a system of manifolds. Show that locally it is diffeomorphic to a system of linear spaces.

(ii) Let Z, W be a system of two submanifolds in a manifold X. Show that if they are transversal then locally this system is diffeomorphic to a system of linear spaces.

2. Let $\nu : X \to Y$ be a morphism of manifolds. Suppose that at all points $x \in X$ the rank of the (linear) tangent map $d\nu$ equals k.

Show that morphism ν is locally diffeomorphic to a linear morphism of linear spaces.

3. Let V be the space of matrices $Mat(m, n)$ of size $m \times n$. For every r consider the subset $M_r \subset V$ of matrices of rank r.

(i) Show that this is a submanifold. Compute its dimension.

(ii) For every point $m \in M_r$ describe the tangent space $T_m(M_r) \subset V$

4. Let $\nu(t) : X \to Y (0 \leq t \leq 1)$ be a smooth homotopy. Show that there exists a smooth homotopy $\mu(t) : X \to Y (0 \leq t \leq 1)$ such that $\mu(t) = \nu(0)$ for $t < 1/4$ and $\mu(t) = \nu(1)$ for $t > 3/4$.

Using this show that smooth homotopy is an equivalence relation on the set of morphisms $\mathrm{M}or(X, Y)$.

5. Let X be a smooth compact manifold of dimension k.

(i) Show that there exists an immersion $\nu : X \to \mathbf{R}^{2k}$

(ii) Show that the exists a morphism $\nu : X \to \mathbf{R}^{2k-1}$ which is an immersion everywhere except finite number of points.

6. Let $\alpha : M \to S$, $\beta : N \to S$ be morphisms of manifolds. We say that they are **transversal** if for any pair of points $x \in M$ and $y \in N$ either $\alpha(x) \neq \beta(y)$ or these points are equal to some point $b \in S$ and we have $D\alpha(T_x M) + D\beta(T_y N) = T_b S$.

(i) Show that this is equivalent to the condition that the product morphism $\alpha \times \beta : M \times N \to S \times S$ is transversal to the diagonal submanifold $\Delta S \subset S \times S$.

(ii) Show that in this case the **fibered product** $W = \{(x, y) \in M \times N | \alpha(x) = \beta(y)\}$ is a submanifold; compute its dimension and describe its tangent space.

7. Let $\nu : X \to Y$ be a morphism of manifolds. Let $N \subset X$ be a compact submanifold such that the restriction of ν to N is an imbedding and for every point $x \in N$ the differential $D\nu : T_x X \to T_{\nu(x)} Y$ is an isomorphism.

(i) Show that the morphism ν defines a diffeomorphism of some neighborhood U of N in X with some neighborhood V of $\nu(N)$ in Y.

(ii) Show that the same statement holds if we do not assume that N is compact, only that N is closed submanifold in X and $\nu : N \to Y$ is a closed imbedding.