Introduction to Differential Geometry

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Glossary

I. SET THEORY.

1. Map. Let X, Y be sets. By definition a map $\nu : X \to Y$ is a correspondence (i.e. a rule) which assigns to every element $x \in X$ some element $\nu(x) \in Y$. Composition of maps $X \to Y$ and $Y \to Z$ is defined as usual.

The set of all maps from X to Y we denote $\mathbf{Maps}(X, Y)$.

Remark. In many books and courses a map $X \to Y$ is called "function". We will not follow this usage since we use the word function in different sense.

2. Function. By definition a function on a set X is a map from X to $\mathbf{R}$ (i.e. function for us means "a real valued function"). The set of all functions we denote by $\mathcal{F}(X)$ (i.e. $\mathcal{F}(X) = \mathbf{Maps}(X, \mathbf{R})$).

Any map $\nu : X \to Y$ defines the **induced** map $\nu^* : \mathcal{F}(Y) \to \mathcal{F}(X)$.

For us important fact is that $\mathcal{F}(X)$ has a natural structure of a commutative algebra and that the map $\nu^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ is a morphism of algebras (for this reason we will call it "morphism ν^*").

II. METRIC SPACES.

1. Metric space. Let X be a set. A **distance function** on X is a function d on $X \times X$ with the following properties:

(i) $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$.

(ii) **Triangular inequality**. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(iii) $d(x, y) = 0$ iff $x = y$.

By definition a **metric space** is a set X equipped with a distance function d.

Remark. If d_1, d_2 are two distance function on X we say that they are **comparable** (notation $d_1 \sim d_2$) if there exists a constant C such that $d_1 \leq Cd_2$ and $d_2 \leq Cd_1$.

In many situations comparable distance functions should be considered as equivalent (for example, they define the same topology).

2. Topology on a metric space. Let (X, d) be a metric space. We use notation $B(a, r)$ for the open ball around a point $a \in X$ of radius r. In other words $B(a, r) = \{x \in X | d(x, a) < r\}$.

We say that a subset $U \subset X$ is a **neighborhood** of a point $a \in X$ if it contains a ball $B(a, r)$ for some $r > 0$.

A subset $U \subset X$ is called **open** if for any point $a \in U$ the set U is a neighborhood of a.

A subset $F \subset X$ is called **closed** if its complement $U = X \setminus F$ is open.

We say that a sequence of points $x_i \in X$ converges to a point $a \in X$ if distances $d(x_i, a)$ tend to 0.

3. Continuous map. Let X, Y be metric spaces. A map $\nu : X \to Y$ is called **continuous at point** $a \in X$ if for any sequence of points $x_i \in X$ which converges to a the sequence of points $\nu(x_i) \in Y$ converges to $\nu(a)$.

The map is called continuous if it is continuous at every point of X.

4. Algebra of continuous functions. If X is a metric space we denote by $C(X)$ the subset of continuous functions in $\mathcal{F}(X)$. It is easy to see that this is a subalgebra.

Every continuous map $\nu : X \to Y$ defines a morphism of algebras $\nu^* : C(Y) \to C(X)$.

III. **ALGEBRA.**

1. Algebra. Algebra A is a vector space A (over $\mathbf{R}$) equipped with a bilinear operation $m : A \times A \to A$. Usually this operation is called multiplication; notation for the image $m(a, b)$ is $a \cdot b$ or simply ab.

The term bilinear means that for any $b \in A$ the maps $a \mapsto ab$ and $a \mapsto ba$ are linear morphisms of vector spaces.

Remark. In fact such object usually is called "algebra over $\mathbf{R}$" or "$\mathbf{R}$-algebra".

Algebras A is called **associative** if it satisfies $a(bc) = (ab)c$ for all $a, b, c \in A$.

We say that the algebra A has a **unit** element if there exists an element $1 \in A$ such that $1a = a = a1$ for all $a \in A$. (Show that such element is uniquely defined if it exists)

We say that the algebra A is commutative if $ab = baforalla, b \in A$.

Usually algebras which we consider are associative, commutative and have 1.

A morphism (or equivalently homomorphism) of algebras is a linear map $r : A \to C$ which preserves multiplication and unit element i.e. $r(ab) = r(a)r(b)$ and $r(1) = 1$.