Problem assignment 1

Introduction to Differential Geometry

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Problems in linear algebra.

1. Let $A : V \to L$ be a morphism of vector spaces, $K = \ker(A)$ its kernel and $I = Im(A)$ its image.
Show that $K \subset V$ and $I \subset L$ are linear subspaces.
Show that A is mono iff $K = 0$. Show that if $K = 0$ and $I = L$ then A is an isomorphism, i.e. there exists an inverse morphism $B : L \to V$ such that compositions $A \circ B$ and $B \circ A$ are identity morphisms.

2. Let V be a vector space and $W \subset V$ a subspace. Show that there exists a vector space Q and an epimorphism $p : V \to Q$ such that $\ker p = W$.
Show that the pair (Q, p) is uniquely defined up to **canonical** isomorphism (i.e. any two such pairs are canonically isomorphic).
The space Q is called the **quotient space**; usually it is denoted by V/W.

3. Let $A : V \to L$ be a linear operator, $K = \ker(A)$ and $I = Im(A)$.
Show that the space I is canonically isomorphic to V/K.

[P] 4. Let V be a finite dimensional vector space of dimension n and $W \subset V$ be a subspace of dimension l.
Show that there exists a basis $e_1, ..., e_n$ of the space V such that the vectors $e_1, ..., e_l$ form a basis of the space W.
Show that in this case the vectors $e_{l+1}, ..., e_n$ (or more precisely their images) form a basis of the quotient space V/W.
Prove that $\dim V/W = \dim V - \dim W$.

[P] 5. Let V be a vector space of dimension n, $L, L' \subset V$ subspaces of V. Show that if $\dim L + \dim L' > n$ then L and L' have a non-zero intersection.

[P] 6. Let $A : V \to L$ be a linear operator between vector spaces. Suppose we know that V and L are finite dimensional vector spaces of the same dimension n.
Show that the following conditions are equivalent:
(i) A is a monomorphism
(ii) A is an epimorphism
(iii) A is an isomorphism

7. Let B be a symmetric bilinear form on finite dimensional vector space V. Denote by Q the corresponding quadratic form on V defined by $Q(x) = B(x, x)$.
(i) Show that the form B could be recovered from Q.
(ii) Show that a function Q on V is a quadratic form iff in any coordinate system it could be written as $\sum a_{ij} x^i x^j$.
[P] (iii) Show that Q is a quadratic form iff it is homogeneous function of degree 2 which for any $a, b \in V$ satisfies the condition that the function $Q(x + a + b) - Q(x + a) - Q(x + b) + Q(x)$ is a constant function.

[P] 8. Let V be a vector space of dimension n and $W \subset V$ a subspace. Consider its orthogonal complement $W^\perp \subset V^*$ defined by $W^\perp := \{f \in V^* | f|W = 0\}$.
(i) What is the dimension of $W^\perp$?
(ii) Show that $(W \cap U)^\perp = W^\perp + U^\perp$ and $(W + U)^\perp = W^\perp \cap U^\perp$.
(iii) Show that $(W^\perp)^\perp = W$.
(iv) Show that $W^\perp$ is naturally isomorphic to $(V/W)^*$.