Problem assignment 2

Introduction to Differential Geometry

Joseph Bernstein November 6, 2006.

Problems about metric spaces.

1. Let $\nu : X \to Y$ be a map of two metric spaces.

(i) Consider a point $x \in X$ and its image $y = \nu(x) \in B$.

Show that ν is continuous at the point x iff it satisfies

(*) For any neighborhood $V \subset Y$ of the point y the subset $\nu^{-1}(V) \subset X$ is a neighborhood of x.

(ii) Show that ν is continuous (i.e. continuous at all points) iff it satisfies

(**) For any open subset $V \subset Y$ the subset $\nu^{-1}(V) \subset X$ is open.

2. Let X be a metric space, $Y \subset X$ a subset of X and $F = Closure(Y)$ its closure (the minimal closed subset containing Y).

Show that a point $a \in X$ belongs to F iff there exists a sequence of points $x_i \in Y$ which converges to the point a.

3. Let X be a metric space. Show that a map $\pi : X \to \mathbf{R}^n$ is continuous iff its coordinate functions $f^i = \pi^i$ are continuous.

4. Let X be a compact metric space.

(i) Show that any closed subset $F \subset X$ is compact (with respect to induced metric).

(ii) Show that for any continuous map $\nu : X \to Y$ from X to a metric space Y the subset $\nu(X) \subset Y$ is closed and compact.

[P] 5. Consider a subset $X \subset \mathbf{R}^n$. Show that X is compact iff it is bounded and closed subset of $\mathbf{R}^n$.

[P] 6. Let $A, B \subset X$ be two non-empty subsets of a metric space X. We define the distance $d(A, B)$ by $d(A, B) = \inf(d(a, b) | a \in A, b \in B)$.

(i) Show that if A is a set consisting of one point a then $d(A, B) = 0$ iff a lies in the closure of the set B.

(ii) Suppose $X = \mathbf{R}^n$, A is closed and B is compact. Show that there exist points $a \in A$ and $b \in B$ such that $d(a, b) = d(A, B)$.

(iii) Construct example of two closed subsets $A, B \subset \mathbf{R}^n$ such that A and B do not intersect but $d(A, B) = 0$.

[P] (*) **7.** This problem gives an equivalent definition of compactness which often is more convenient than the original definition.

Let X be a metric space. Show that it is compact iff it satisfies the following

Finite Covering Property. Any open covering $\{U_\alpha\}$ of the space X contains a finite subcovering $\{U_{\alpha_i}\}$.

[P] 8. Let X be a compact metric space and $\{F_\alpha\}$ a family of closed subsets of X .

Suppose we know that any finite collection of these subsets has non-empty intersection. Show that all these subsets have non-empty intersection.

[P] 9. Let C be a compact subset of a metric space X and $U \subset X$ be an open subset which contains C. Show that there exists $\varepsilon > 0$ such that U contains ε -neighborhood of C.

[P] 10. (i) Find the maximal area of a triangle inscribed into a unit circle.

(ii) Can you describe how to evaluate the maximal area of a convex 10-gon inscribed into the unit circle.