Problem assignment 3

Introduction to Differential Geometry

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Extension of smooth functions.

[P] 1. Cut off functions. Fix number r such that $0 < r < 1$.

(i) Show that there exists a smooth function $u(t)$ in one variable such that $u(t) = 0$ for $|t| > 1$ and $u(t) = 1$ for $|t| \leq r$.

(ii) Fix a point $a \in \mathbf{R}^n$. Consider two concentric balls $B(a,r), B(a,R)$ where $r < R$. Show that there exists a smooth non-negative function $v(x)$ on $\mathbf{R}^n$ which vanishes outside of the ball $B(a,R)$ and is equal to 1 inside the ball $B(a,r)$.

Such function is called a **cut-off function**.

Hint. Show that the function $a(t)$ in one variable defined by $a(t) = 0$ for $t \leq 0$ and $a(t) = \exp(-1/t^2)$ for $t > 0$ is smooth.

Construct a smooth non-negative function $b(t)$ such that $b(t) = 0$ for $|t| > 1$ and $b(0) = 1$.

Construct a smooth non-negative function $c(t)$ such that $c(t) = 0$ for $t \leq 0$ and $c(t) = 1$ for $t \geq 1$.

[P] 2. Extension of smooth functions. (i) Let f be a smooth function on the ball $B(a,R)$ of radius R in $\mathbf{R}^n$, and let $B(a,r)$ be a smaller ball (i.e. $r < R$).

Show that one can extend the function f from the smaller ball $B(a,r)$ to the whole space $\mathbf{R}^n$ as a smooth function. Note that it is not always possible to extend f from the larger ball $B(a,R)$ as a smooth function.

(*/2) (ii) Consider a domain U and its open subdomain W. Suppose we are given a smooth function f defined on W. Show that for every compact subset $C \subset W$ we can find a smooth function h on U which coincides with f on some neighborhood of C.

Variations on inverse function Theorem.

Let $\pi : X \to Y$ be a morphism of smooth domains $X \subset V$ to $Y \subset L$, $a \in X, b = \pi(a) \in Y$. Denote by $A : V \to L$ the differential $D\pi_a$.

Fix coordinate systems $x^1, ..., x^m$ and $y^1, ..., y^n$ on X and Y and denote by $J(x)$ Jacobi matrix of π (this is a matrix with values in $S(X)$).

3. Suppose $m = n$ and the operator A is invertible (i.e the matrix $J(a)$ is invertible). Show that π is an isomorphism (i.e. diffeomorphism) of some neighborhood U of $a \in X$ to some neighborhood W of $b \in Y$.

[P] 4. Suppose that the morphism π is **submersive at the point** a which means that the operator A is epimorphism, i.e. Jacobi matrix has rank n .

Show that one can choose new coordinate system on X near a so that in new coordinates the morphism π takes the standard form $\pi^*(y^j) = x^j$ for $j = 1, ..., n$.

[P] 5. Suppose that the morphism π is **immersion at the point** a which means that the operator A is monomorphic, i.e. Jacobi matrix has rank m.

Show that in this case one can change coordinate system on Y near point b such that in the new system the morphism π takes the standard form $\pi(x^1, ..., x^m) = (x^1, ..., x^m, 0...0)$.

[P] 6. Consider a smooth function $f(x,y)$ defined in a neighborhood of a point $a = (0,0)$. Suppose that $f_x(a) = 0$ and $f_{xx}(a) = 1$.

Show that if we choose a small neighborhood U of a then for small y the function $u(y) = \min\{f(x,y) | (x,y) \in U\}$ is a smooth function.

Can you compute the derivative of u at the point $y = 0$?

7. Let $u(t)$ be smooth function which vanishes for $t \leq 0$ and is positive for $t > 0$. Consider the following functions h, f in two variables

$h(x,y) = u(3x^2 - y) \cdot u(y - x^2)$

$f(x,y) = h(x,y)/h(x, 2x^2)$, $f(0,0) = 0$.

Show that the restriction of the function f to any straight line is smooth but f is not a continuous function on $\mathbf{R}^2$.