Introduction to Differential Geometry

Sample problems for preparation to the final exam.

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1. Let X be a domain of dimension n, a a point of X. Consider two submanifolds M, N of X containing the point a. We say that M and N intersect **transversally** at the point a if the tangent spaces T_aM and T_aN generate the space T_aX.

(i) Show that in this case the subset $L = M \bigcap N \subset X$ is a submanifold in some neighborhood of the point a. Describe the tangent space T_aL.

(ii) Give example of two surfaces in $\mathbf{R}^3$ which intersect not transversally such that their intersection is not a manifold.

2. Let $C \subset \mathbf{R}^2$ be a curve defined by equation $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ (i.e. C is an ellipse with half axes a, b).

Compute the curvature of C at all points.

3. Let M be a compact manifold; we fix an orientation μ on M. Let ξ be a vector field on M and ω a top degree form on M.

Show that the integral $\int_M Lie_\xi(\omega)$ is equal to 0.

4. Let X be a domain of dimension 3 . Consider differential forms $\alpha \in \Omega^1(X)$ and $\rho \in \Omega^2(X)$.

Suppose we would like to find a differential 1-form β on X such that $\rho = \alpha\beta$.

Show that for this there is a necessary condition. namely the condition (*) $\alpha\rho = 0$.

Show that if at every point x of the domain X the form α if not 0 then the necessary condition (*) is in fact sufficient for the existence of the form β.

5. Let N be a manifold and $\omega \in \Omega^k(N)$ a closed differential form on N. Let M be another manifold and $\nu_i : M \to N$, $i = 1, 2$, two morphisms of manifolds. Consider the inverse images $\rho_i = \nu_i^*(\omega) \in \Omega^k(M)$.

Show that if morphisms ν_1, ν_2 are homotopic then the forms ρ_1, ρ_2 are cohomologues, i.e. their difference is a differntial of some form $\eta \in \Omega^{k-1}(M)$.

6. Let M be a Riemannian manifold. For any smooth function f on M we denote by $grad_f$ its gradient vector field on M.

Show that if f, h are two smooth functions on M then $grad_f(h) = grad_h(f)$.

7. Consider the curve $c(t) = (t, t^2, t^3)$ in $\mathbf{R}^3$. Compute the curvature of this curve.

8. Let M be a compact Riemannian manifold and ω a k-form on M. We would like to investigate the form ω by considering its integrals $\int_\nu \omega$ over cycles $\nu : S \to M$, where $S = S^k$ is the standard k-dimensional sphere considered with the standard orientation.

Let us say that the size of the cycle ν is $\leq r$ if the differential $D\nu$ has norm $\leq r$ at all points of the sphere S. This allows us to define the norm $d(\nu)$ of the cycle ν to be the minimum of all such numbers r.

(i) Show that $\int_\nu \omega$ is $O(d(\nu)^{k+1})$ when $d(\nu)$ tends to 0.

(ii) Suppose we know that the stronger estimate holds $\int_\nu \omega = o(d(\nu)^{k+1})$.

What can you tell about the form ω ?

What can you tell about the integrals $\int_\nu \omega$ when $d(\nu)$ is very small ?

9. Let M be a Riemannian manifold with Riemann metric B, $L \subset M$ a submanifold of M and C the induced Riemann metric on L.

Fix a smooth function f on M and denote by h its restriction to the submanifold L.

Show that for any point $a \in L$ we have inequality

$$||grad_B f|| \geq ||grad_C h||$$

10. Consider the plane curve C which in polar coordinates (r,ϕ) is given by equation $r = 5\phi$ (it is called a spiral).

Compute the curvature of the curve C at all points.

11. Prove Green's theorem. Let M be the Euclidean space $\mathbf{R}^n$ with the standard metric, $D \subset M$ a domain with smooth boundary ∂D and f a smooth function on D.

Then the flow of the vector field $\xi = gradf$ through the hypersurface ∂D equals to the integral over D of the function $h = \Delta f$, where $\Delta = \sum(\partial_i)^2$ is the Laplace operator.

12. Let $M \in \mathbf{R}^3$ be the paraboloid given by the equation $z = x^2 + y^2$. We consider the Riemannian metric B on M induced from the standard Riemannian metric on $\mathbf{R}^3$.

Fix the system of coordinates (x, y) on M.

(i) Write the metric B in this coordinate system.

(ii) For any smooth function f on M write its gradient vector field (with respect to the metric B).

(iii) Write explicitly first and second fundamental forms on M. Compute the Gaussian curvature of M.

13. Let X be a manifold and f, h two smooth functions on X. We would like to find a vector field ξ on X such that $\xi(f) = h$.

Show that if we can do this locally on X then we can do this globally on X.

14. Consider a curve $C \subset \mathbf{R}^2$ defined by equation $f(x, y) = 0$. Fix a point $a \in C$ and denote by L the tangent space $T_a(\mathbf{R}^2)$ canonically isomorphic to $\mathbf{R}^2$.

Consider the following quadratic forms on the space L:

form H equal to the Hessian of the function f at the point a
standard Euclidean form B
form $D = (d_a f)^2$.

Show that when a parameter t tends to ∞ the ratio $\det(H+tD)/\det(B+tD)$ converges to the curvature of the curve C at the point a.

15. Similarly to the problem 14 consider a surface $\Sigma \subset \mathbf{R}^3$ defined by equation $f(x, y, z) = 0$.

Construct quadratic forms H, B, D as before and show that when t tends to ∞ the ratio $\det(H + tD)/\det(B + tD)$ converges to the Gauss curvature of Σ.