Take Home Exam.

Representations of Finite Groups.

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This is a take-home exam. You are supposed to solve the problems during 10 days. In other words you have to send me the solution to the exam no later than August 31, 2007.

You should send the solution to me via email to the address bernstei@post.tau.ac.il. You can send it as a tex file or as a scan.

You are encouraged to think about all the problems.

If you have an idea how to solve some problem but do not have complete solution please write your idea.

Usually correct solution of 8 out of 12 problems is enough to get 100 points.

There is a formal requirement for this take-home exam.

You should include in the exam the statement that you did the exam by yourself, without outside help.

Good luck.

In what follows G is a finite group, (π, G, V) a finite dimensional representation of G.

1. (i) Let (π, G, V) be a representation of G over the field $\mathbf{Q}$ of rational numbers. Show that π is isomorphic to the dual representation π^*.

(ii) Show that the same holds true if π is a complex representation which character is defined over $\mathbf{Q}$.

2. Let A be an algebra and M an A-module. Let us denote by $E(M)$ the algebra $\mathcal{H}om_A(M, M)$ of endomorphisms of A-module M.

Show that if the module M is simple, then $E(M)$ is a division algebra.

3. Let G be a symmetric group S_n. Show that any complex representation (π, G, V) can be realized over the field $\mathbf{Q}$ of rational numbers.

4. Fix an element $x \in G$.

(i) Suppose we know that for all irreducible representations (π, G, V) we have $ch_\pi(x) = ch_\pi(e)$, where e is the unit element of G. Show that $x = e$.

(ii) Suppose we know that for all irreducible representations (π, G, V) we have $|ch_\pi(x)| = |ch_\pi(e)|$. Show that x lies in the center of G.

5. Let (π, G, V) be an irreducible representation of G. Consider a subfield $K \subset \mathbf{C}$ which contains all values of the function ch_π.

(i) Show by example that the representation (π, G, V) not always can be defined over the field K.

(ii) Suppose we know that for some subgroup $H \subset G$ the space V^H of H-invariants in V is one dimensional. Show that in this case the representation (π, G, V) can be defined over K.

6. Let X be a G-set with more than one element. Suppose we know that the action of G on X is 2-transitive (i.e. G acts transitively on $X \times X \setminus \triangle(X)$).

Show that the natural representation π_X of G in the space $\mathcal{F}(X)$ is a sum of a trivial representation and an irreducible representation.

7. Let (π, G, V) be a representation defined over the field $\mathbf{R}$ of real numbers. Let us denote by $Q(V)$ the space of G-invariant quadratic forms on V.

(i) Show that if $V \neq 0$ then $Q(V) \neq 0$.

(ii) Show that (π, G, V) is irreducible (over $\mathbf{R}$) iff the space $Q(V)$ is one dimensional.

8. Let us fix a subgroup $H \subset G$ and an irreducible representation ρ of H. For any representation (π, G, V) we denote by $[\pi : \rho]$ the multiplicity of the representation ρ in the restriction $\pi|_H$.

Show that the sum $\sum_{\pi \in Irr(G)} [\pi : \rho]$ is bounded by the index $[G : H]$.

9. An irreducible representation ρ of the group G over the field $\mathbf{R}$ of real numbers can be of one of 3 types - real, complex or quaternionic (this means that the endomorphism algebra $\mathcal{E}(\rho)$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{Q}$).

We denote by N_r, N_c, N_q the number of corresponding isomorphism classes of irreducible representations of the group G over $\mathbf{R}$.

Show that the number $N_r + N_q + 2N_c$ equals to the number of conjugacy classes of G.

10. Let (ρ, G, L) be a finite dimensional representation of G over a finite field F. Consider the dual representation (ρ^*, G, L^*).

Show that the number of G-orbits in L equals to the number of G-orbits in L^*.

11. Let τ be a representations of G. Tensor product with τ defines the operator on the representation ring $R(G)$. In the basis of irreducible representations this operator is given by the matrix of multiplicities $m_{\pi\rho} := [\pi \otimes \tau : \rho]$ for $\pi, \rho \in Irr(G)$.

(i) Suppose we know that $\dim \tau = d$. Show that every number $m_{\pi\rho}$ is bounded by d.

(ii) Show that for a fixed π we have a bound $\sum_\rho m_{\pi\rho}^2 \leq d^2$.

12. Let B_n be the group of motions of an n-cube.

In more details, consider the subset $C \subset \mathbf{R}^n$ consisting of vectors $\pm e_i$, where (e_i) is the standard basis of $\mathbf{R}^n$. Then B_n is the group of linear invertible transformations of $\mathbf{R}^n$ which map the subset C into itself.

This group is similar to the permutation group S_n (sometimes the group B_n is called the signed permutation group).

Using classification theory of representations of symmetric groups describe how to classify the irreducible representations of the group B_n.