Problem assignment 1.

Representations of Finite Groups.

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A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Dmitry).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign $(\Box)$marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

Groups and G-sets.

1. Let G be a group, i.e. a set with three operations
$m : G \times G \to G$, $e : pt \to G$ and $inv : G \to G$ satisfying group axioms.
Show that the multiplication map m uniquely determines the map e.
Show that the map m uniquely determines the map inv.

Definition. Let X be a G-set.

(i) We say that X is **homogeneous** if for any $x, y \in X$ there exists an element $g \in G$ such that $gx = y$.

(ii) We say that X is **free** if stabilizer of every point is trivial (i.e. has one element e).

(iii) We say that X is G-torser is it is non-empty, homogeneous and free.

2. Let $\gamma : G \times X \to X$ be an action of the group G on a set X. It is often convenient to describe the action γ in terms of the **graph map** $\delta : G \times X \to X \times X$ defined by $\delta(g, x) = (x, gx)$.

Show that the action γ is homogeneous iff δ is epimorphic and γ is free iff δ is monomorphic.

[**P**] **3.** Let X be a G-torser. Show that any morphism of G-sets $\nu : X \to X$ is an isomorphism.

Denote by $A = A_X$ the group $Aut(X)$ of all automorphisms of the G-set X. Show that this group is isomorphic to G.

Pay attention that this isomorphism is **not canonical -** it is only canonically defined up to inner automorphisms.

Let X be a G-set. Denote by $A = A_X$ be the group of automorphisms of X (i.e. the group of isomorphisms of the G-set X to itself).

A homogeneous G-set is called **normal** if the group A_X acts transitively on the set X.

[**P**] **4.** Show that if X is homogeneous then the action of the group A_X on the set X is free.

Describe explicitly the group A_X for the case when $X = G/H$. In particular, show that the homogeneous G-set X is normal iff stabilizer subgroups of all points coincide (which means that this subgroup H is a normal subgroup of G).

Definition. Let G be a group and $H \subset G$ a subgroup. The number

$[G:H] := \#(G/H)$ is called the **index** of the subgroup H (it is a natural number or ∞).

[P] 5. (i) Show that if we have a tower of subgroups $F \subset H \subset G$ then we have a product formula $[G:F] = [G:H][H:F]$.

(ii) Let $H \subset G$ be a subgroup of finite index n. Show that H contains a subgroup N which is normal in G such that $[G:N] \leq n!$.

[P] 6. Fix a group G. Subgroups $H, H' \subset G$ are called **commensurable** if there exists a subgroup D which lies inside H and H' and has finite index in both subgroups.

Show that this is an equivalence relation.

Fix one class C of commensurable subgroups. Show that we can uniquely define a function of relative order $[H:F] \in \mathbf{Q}^+$ on all subgroups in class C which satisfies the product formula and coincides with the index $[H:F]$ if H contains F.

[P] 7. Let p be a prime number and S be a finite p -group, i.e. $\#S = p^r$.

(i) Show that if S is non-trivial then its center $Z(S)$ is non-trivial.

(ii) Show that there exists a tower of subgroups $S_0 \subset S_1 \subset S_2 \subset ... \subset S_r = S$ such that $\#S_i = p^i$. Show that we can choose all the subgroups S_i to be normal in S.

(iii) Let G be a finite group of order n. Suppose we know that the number n has a divisor q which is a power of a prime number. Show that in this case G has a subgroup of order q.

Representation Theory.

[P] 8. Let (ρ_1, V_1), (ρ_2, V_2) be irreducible representations of a group G. Consider the direct sum (ρ, V) of these representations.

The space V has four G-invariant "coordinate" subspaces $0, V_1, V_2, V$.

Show that the representations ρ_1 and ρ_2 are equivalent iff there exists a non-coordinate G-invariant subspace in V (i.e. a subspace distinct from the four subspaces listed above).

($\Box$)**9.** Let (ρ, V) be an irreducible complex representation of some group G. Suppose we know that the space V has countable dimension. Show that Schur's lemma holds for ρ.

(**Hint.** Prove and use the fact that any operator $A : V \to V$ has a non-empty spectrum, i.e. there exists $\lambda \in \mathbf{C}$ such that the operator $A - \lambda$ is not invertible).