**Problem assignment 2.**

**Representations of Finite Groups.**

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**[P] 1.** Prove the following

**Theorem**(Maschke). Let $G$ be a finite group of order $n$ and $k$ a field of characteristic prime to $n$ ( i.e. $k$ contains an element $1/n$).

Show that any representation of the group $G$ over the field $k$ is completely reducible.

**[P] 2.** Let $G$ be an infinite group and $H \subset G$ a subgroup of finite index.

Let $(\pi, G, V)$ be a complex representation of $G$ and $L \subset V$ a $G$-invariant subspace.

Suppose we know that the subspace $L$ has an $H$-invariant complement. Show that then it has a $G$-invariant complement.

If $X$ is a finite $G$-set we denote by $\pi_X$ the natural representation of the group $G$ on the space $\mathcal{F}(X)$ of functions on $X$ (check that you know how to write explicit formulas).

**3.** Show that if $X, Y$ are finite $G$-sets then the intertwining number $< \pi_X, \pi_Y >$ equals to the number of $G$-orbits in the set $X \times Y$ (with respect to the diagonal action $g(x, y) = (gx, gy)$).

**[P] 4.**. Let $G = S_n$ denote the group of all permutations of a set $I$ with $n$ elements (symmetric group in $n$ symbols). Consider the natural action of $G$ on the set $X$ of all subsets of $I$. It has orbits $X_0, X_1, ..., X_n$, where $X_i$ consists of all subsets of size $i$ (note that $X_0$ has one element).

Consider the corresponding representations $\pi_i := (\pi_{X_i}, G, \mathcal{F}(X_i))$ of the group $G$ and decompose them into irreducible components.

Describe these components for representations $\pi_0, \pi_1, \pi_2$ and compute their dimensions. Try to do this for the representation $\pi_3$.

Here we assume that $n$ is large, say $n > 6$.

## Algebras and modules.

In future assignments we will fix a field $k$. By $k$-algebra $A$ we mean an associative $k$-algebra $A$ with 1. We denote by $\mathcal{M} = \mathcal{M}(A)$ the category of (left) $A$-modules.

An $A$-module $M$ is called **simple** if it is non-zero and has no submodules different from 0 and $M$.

**5.** Let $M$ be an $A$-module. Consider the space of endomorphisms $\mathcal{E}(M) = \mathrm{End}_{\mathcal{M}}(M)$.

Show that $\mathcal{E}(M)$ has a structure of $k$-algebra.

Show that if $M$ is simple then $\mathcal{E}(M)$ is a division algebra.

Show that if $M = A$ is a free module with one generator then $\mathcal{E}(M)$ is isomorphic to the opposite algebra $A^o$.

**6.** Let $G$ be a group. Consider the group algebra $\mathcal{H}(G) = \mathcal{H}(G; k)$. By definition this is a $k$-vector space with the basis $\{\delta_g | g \in G\}$ with multiplication given by the formula $\delta_g \delta_h = \delta_{gh}$.

Convince yourself that the category $\mathcal{M}(\mathcal{H}(G))$ is canonically the same as the category of representations of the group $G$ over the field $k$.

**Definition**. Let $M$ be an $A$-module and $L \subset M$ a submodule. We say that $L$ **splits as a direct summand in** $M$ if there exists a complementary $A$-submodule $R \subset M$ (this means that $L \cap R = 0$ and $L + R = M$).

We say that an $A$-module $M$ is **completely reducible** is any submodule in $M$ splits as a direct summand.

**[P] 7.** Show that for a submodule $L \subset M$ the following conditions are equivalent:

(i) $L$ splits in $M$ as a direct summand

(ii) Let $i : L \to M$ denote the natural imbedding. Then there exists a left inverse morphism of $A$-modules $p : M \to L$ (this means that $p \circ i = Id_L$).

(iii) For any module $N \in \mathcal{M}(A)$ the natural restriction map $Hom_{\mathcal{M}}(M, N) \to Hom_{\mathcal{M}}(L, N)$ given by $\nu \mapsto \nu \circ i$ is an epimorphism of sets.

**Remark.** The last condition (iii) is a typical way how different properties of objects and morphism are characterized in category theory - not in term of inner structure, but in term of relations to other objects.

**8.** (i) Show that if $A$-module $M$ is completely reducible then any subquotient of $M$ is also completely reducible.

(□)(*) (ii) Show that if $A$-modules $M, N$ are completely reducible then their direct sum $M \oplus N$ is also completely reducible.

**[P] 9.** Assume that the field $k$ is algebraically closed and $A$-module $M$ has finite dimension over $k$. We would like to understand the structure of the algebra $\mathcal{E}(M) = \mathrm{End}_{\mathcal{M}}(M)$.

(i) Show that if $M$ is simple then $\mathcal{E}(M) = k$.

(ii) Show that if $M$ is completely reducible then $\mathcal{E}(M)$ is isomorphic to a finite direct sum of matrix algebras over $k$.

(□)**10.** Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Let us consider finite dimensional $A$-modules.

Show that the following conditions are equivalent

(i) Any $A$-module is completely reducible.

(ii) Free module $A$ is completely reducible.

(iii) $A$ has a faithful completely reducible module $M$.

(iv) $A$ is isomorphic to a direct sum of matrix algebras.

**Remark.** The meaning of the markings for the problems is explained in the assignment 1.