Problem assignment 3.

Representations of Finite Groups.

Joseph Bernstein March 15, 2007.

[P] 1. Let G be a finite group and let π be a finite dimensional representation of G over the field of real numbers $\mathbf{R}$.

(i) Show that π is isomorphic to the dual representation π^*.

(ii) Suppose that the group G is commutative and the representation π is irreducible. Show that $\dim \pi$ equals 1 or 2 and show that both cases are possible.

(iii) Give an example of irreducible representations (π, G, V) and (ρ, H, L) over the field $\mathbf{R}$ such that the tensor product representation $(\pi \otimes \rho, G \times H, V \otimes L)$ is reducible.

[P] 2. Let (π, V) be an irreducible complex representation of G. Show that it has an invariant Hermitian form H and that any two such forms are proportional.

[P] 3. Let C be a finite commutative group and D a subgroup of C.

Show that any character $\chi : D \to \mathbf{C}^*$ of the group D can be extended to some character of the group C.

Fourier transform for finite groups.

Let C be a finite commutative group, $n = \#(C)$. We will denote by $\check{C}$ the dual group of characters $\psi : C \to \mathbf{C}^*$.

We define **Fourier transform** $F : \mathcal{F}(C) \to \mathcal{F}(\check{C})$ by $F(u)(\psi) = \sum u(g)\psi(g)$.

[P] 4. Show that if we define an L^2-structure on spaces of functions by $||u||^2 = 1/n \sum |u(g)|^2$ then the operator F satisfies the Plancherel formula $||F(u)||^2 = n||u||^2$.

Using this write down the explicit inversion formula for F.

Let $\mathbf{F}$ be a finite field of order q. Fix a non-trivial additive character $\psi_0 : \mathbf{F} \to \mathbf{C}^*$.

Let V be a finite dimensional vector space over $\mathbf{F}$. Then its dual group can be described explicitly.

Namely consider a homomorphism $\nu : V^* \to \check{V}$ defined by $\nu(\xi)(v) = \psi_0(\xi(v))$. Show that ν is an isomorphism of groups.

Usually we will identify groups V^* and $\check{V}$ using this isomorphism.

Consider the multiplicative group $\mathbf{F}^*$. It naturally acts on $\mathbf{F}$ (by multiplication) and hence on the space $L = \mathcal{F}(\mathbf{F})$.

[P] 5. (i) Describe explicitly the decomposition of this space into irreducible components. Namely for any multiplicative character χ (i.e. a character of the multiplicative group $\mathbf{F}^*$) describe explicitly the subspace $L_\chi \subset L$ of functions which transform according to χ.

(ii) Show that the Fourier transform F maps L_χ into $L_{\chi^{-1}}$.

(iii) Using the Plancherel formula prove the following

Theorem(Gauss). Fix a non-trivial multiplicative character χ and a non-trivial additive character ψ for the field $\mathbf{F}$ and consider the **Gauss sum**

$G = \sum \chi(g)\psi(g)$, where the sum is taken over $g \in \mathbf{F}^*$.

Then $|G| = q^{1/2}$.

Grothendieck group.

Definition. Let C be a category which has some notion of subobject and quotient object. We define the **Grothendieck group** $K(C)$ as the abelian group generated by symbols $[X]$ for isomorphism classes of objects $X \in Ob(C)$ and relations $[X]-[Y]-[X/Y]=0$ for isomorphism classes of pairs $Y \subset X$ with $X, Y \in Ob(C)$.

[P] 6. Fix a field k. Compute the Grothendieck group $K(C)$ of the category C in the following cases:

(i) C is the category of finite dimensional vector spaces over k.
(ii) C is the category of all vector spaces over k.
(iii) C is the category of finitely generated modules over the algebra $A = k[x]$.
(iv) C is the category of A-modules which are finite dimensional over k (you can assume k to be algebraically closed).

7. Let G be a finite group and C the category of complex finite-dimensional representations of G.

Describe the Grothendieck group $K(C)$ and describe the structure of a ring on this group (this ring is usually called **representation ring** of G and denoted $R(G)$).

Characters.

Let A be an algebra over an algebraically closed field k.

For any finite-dimensional A-module M define its **character** $ch_M \in A^*$ by formula $ch_M(a) = tr(a|M)$.

($\square$)**8.** Show that if finite-dimensional modules $L_1, ...L_k$ represent distinct elements of $Irr(A)$ (the set of isomorphism classes of simple A-modules) then the functionals ch_{L_i} are linearly independent elements of A^*.

Hint. Define an action of the algebra A on the space A^* and show that the submodule N_L generated by a character ch_L is a direct sum of simple modules isomorphic to L.

Finite dimensional algebras.

Definition. Define **radical** of A to be the subset $R \subset A$ of all elements $a \in A$ which acts as 0 in any irreducible (i.e. simple) A-module M.

($\square$)**9.** Suppose that the algebra A is finite-dimensional.

Show that R is a two sided ideal of A.

Show that R is nilpotent, i.e. there exists a constant l such that $R^l = 0$.

Show that any nilpotent two sided ideal $J \subset A$ lies inside the radical. In other words, the radical R can be defined as the maximal two sided nilpotent ideal.

Show that the algebra $B = A/R$ is isomorphic to a direct sum of matrix algebras (such algebra is call **semi-simple**).