Problem assignment 5.

Representations of Finite Groups.

Joseph Bernstein April 19, 2007.

Let G be a finite group and $N \subset G$ a normal subgroup. The group G acts on N via conjugation and hence it acts on the set $I = Irr(N)$.

Definition. A G-equivariant sheaf F on the set I is called **special** if for any point $\rho \in I$ the action of the group N on the fiber F_ρ is isotypical of type ρ.

1. Show that the full subcategory of $Sh_G(I)$ consisting of special sheaves is naturally equivalent to the category $Rep(G)$.

2. Let (ω, G, V) be an irreducible representation of G. Suppose we know that the restriction $\omega|_N$ is not an isotypical representation.

Show that in this case ω is induced, i.e. there exists a subgroup $H \subsetneqq G$ containing N and an irreducible representation σ of H such that ω is isomorphic to $Ind_H^G(\sigma)$.

Definition. Let us call a group G **c-solvable**(which means cyclicly solvable) if there exists a sequence of normal subgroups $N_0 \subset N_1 \subset ... \subset N_k = G$ starting with the trivial subgroup N_0 such that each quotient group N_i/N_{i-1} is cyclic.

3. Show that any subgroup and quotient group of a c-solvable group is c-solvable.

Show that any finite nilpotent group is c-solvable.

[P] 4. (i) Let G be a c-solvable finite group. Show that any irreducible representation σ of G is induced from a one dimensional representation of some subgroup.

(ii) Suppose we know that a group G has a commutative normal subgroup N such that the group G/N is c-solvable. Show that any irreducible representation σ of G is induced from a one dimensional representation of some subgroup.

[P] 5. Let G be a finite group, D its subgroup and χ a character of D. Consider the induced representation $\pi = Ind_D^G(\chi)$.

Show that π is irreducible iff the following condition holds:

(*) For any $g \in G \setminus D$ there exists an element $x \in D$ such that the element $y = gxg^{-1}$ belongs to D and $\chi(x) \neq \chi(y)$.

[P] 6. Let G be a finite group, Z its central subgroup and χ a character of Z. Denote by $Irr(G)_\chi$ the set of equivalence classes of irreducible representations of G with the central character χ.

(i) Compute $\sum dim(\rho)^2$, sum over all elements in $Irr(G)_\chi$.

(ii) Explain how to find the size of the set $Irr(G)_\chi$.

In particular show that this size is maximal when χ is a trivial character.

[P] 7. Let γ be an action of a finite group G on a finite abelian group C. Denote by δ the induced action of the group G on the dual group $\check{C}$.

Show that the number of G-orbits in C equals to the number of G-orbits in $\check{C}$.

Apply this to the case when C is a vector space over a finite field F - formulate the resulting statement.