Problem assignment 8.

Representations of Finite Groups.

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Consider a finite field $F = \mathbf{F}_q$. To simplify the computations you can assume that the size q of the field F is "large".

We consider groups $G = SL(2, F)$ and $G = GL(n, F)$ and describe a method which in principle allows to obtain the classification of irreducible representations of these groups. In order to do this we will recall some basic definitions and properties of algebraic groups and their representations.

Fix $K = \bar{F}$ an algebraic closure of F.

Definition. (i) An **algebraic group** G is an algebraic variety G equipped with a structure of a group, such that the group operations m (the multiplication map) and inv (the inverse map) are morphisms of algebraic varieties.

The basic algebraic group is the group $GL(n, K)$. We will consider only groups G which are subgroups of $GL(n, K)$ closed in Zariski topology (i.e. they are defined by a system of polynomial equations).

(ii) An algebraic representation π of an algebraic group G is a finite dimensional representation over the field K such that the corresponding map $\pi : G \to GL(V)$ is an algebraic morphism.

For an algebraic group G denote by $Rep(G)$ the category of algebraic representations of G and by $R(G)$ its Grothendieck group.

Consider the algebraic group $K^* = GL(1, K)$. For this group there is a standard convenient notation G_m - multiplicative group. We will use this notation.

1. (i) Show that any algebraic group homomorphism $\rho : G_m \to K^*$ has a form $a \mapsto a^n$ for some integer n.

In other words the group $L(G_m) = Mor(G_m, K^*)$ of algebraic morphisms from G_m to K^* is canonically isomorphic to $\mathbf{Z}$.

(ii) Show that the ring $R(G_m)$ is isomorphic to the group ring $\mathbf{Z}(L(G_m))$ of the group $L(G_m)$ (this is a free abelian group with basis given by elements of the group $L(G_m)$ and natural multiplication law).

[P] 2. More generally, let us consider the group $H = H_n$ of diagonal matrices in $GL(n, K)$ (we consider it as an algebraic group).

(i) Show that the group $L(H) = Mor(H, K^*)$ is naturally isomorphic to $\mathbf{Z}^n$.

(ii) Show that the representation ring $R(H)$ is naturally isomorphic to the group algebra $\mathbf{Z}(L(H))$ of the group $L(H)$.

We would like to study (algebraic) representations of the group $G_n = GL(n, K)$. Usually we can get a lot of information about representations of G_n by restricting them to the subgroup of diagonal matrices $H = H_n$.

[P] 3. Consider the natural restriction morphism $res : R(G_n) \to R(H_n)$.

(i) Show that the image R of the restriction morphism is invariant with respect to the natural action of the symmetric group $W = S^n$ on $L(H_n)$ and hence on $R(H_n)$.

(ii) Show that the image R contains all W-invariant elements

(*) (iii) Show that res is an imbedding (we will not use this fact).

Brauer character construction..

Fix a W-invariant finite subset $O \in L(H_n)$ (you can think about O as a W-orbit). Starting from such subset O we will construct in a very explicit and simple way a complex valued function f_O on $GL(n, K)$ which has a property that its restriction to any finite subgroup $D \subset GL(n, K)$ is a character.

First of all fix a nondegenerate character (i.e. an imbedding) $\nu : K^* \to \mathbf{C}^*$.

Suppose we are given an element $g \in GL(n, K)$. We will assign to g an element $h(g) \in H_n$ defined up to permutations (i.e. up to the action of the symmetric group W on H_n) .

Namely, $h(g)$ is an element conjugate to the semisimple part of g. In other words, one can think about $h(g)$ as a collection of eigenvalues of the matrix g.

Now we set $f_O(g) = \sum_{\xi \in O} \nu(\xi(h(g)))$.

Since the set O is W-invariant this number does not depend on the choice of the representative $h(g)$.

[P] 4. Show that for any finite subgroup $G \subset GL(n, K)$ the restriction of the function f_O to G lies in $R(G) = \mathcal{C}h(G)$.

Hint. Use the result of problem 3 to construct an element $r \in R(GL(n, K))$ such that the function f_O coincides with the Brauer character corresponding to the element r.

[P] 5. (i) Use the result of problem 4 in order to classify irreducible representations of the group $GL(2, F)$.

Try to do the same for $GL(3, F)$, $GL(4, F)$.

(ii) Classify irreducible representations of the group $SL(2, F)$ when characteristic of F is odd.

(□)**6.** Consider the group $G = GL(n, F)$.

(i) Describe "most" of irreducible representations of the group G.

(*) (ii) Classify all irreducible representations of G.

Let us discuss some generalizations of Jordan-Hoelder theorem.

Fix an algebra A. We will study the category $\mathcal{M}$ of (left) A-modules. Let us denote by I the set of isomorphism classes of objects in $\mathcal{M}$.

Definition. (i) Let M be an A-module. A **filtration** of M is a finite decreasing sequence Φ of submodules $M = M_0 \supsetneqq M_1 \supsetneqq ... \supsetneqq M_k = 0$.

(ii) Given a filtration Φ we denote by $Cont(\Phi)$ (content of Φ) the multiset of elements in I consisting of all quotients M_i/M_{i-1} (we count them with multiplicities, but do nor care about their order).

In other words, one can think about $Cont(\Phi)$ as a function on I which assigns to each isomorphism class of modules R the multiplicity $m(R)$ which is the number of places when R appears in the sequence of quotients M_i/M_{i-1}.

(iii) We say that a filtration Ψ **refines** filtration Φ if all submodules which appear in Φ also appear in Ψ.

(□)**7. (Jordan Hoelder Theorem).**

(i) Let M be an A-module and Φ, Ψ two filtrations of M. Then there exist their refinements Φ', Ψ' such that $Cont(\Phi') = Cont(\Psi')$.

(ii) Let M, N be two A-modules. We say that they are JH-equivalent if they have filtrations with equal content.

Show that this defines an equivalence relation on the set I.

Show that this is the minimal equivalence relation $\sim$ on the set I which satisfies the following property (semi-simplification property)

(*) If $L \subset M$ then $M \sim (L \oplus M/L)$.

Remark. Explain how each of the statements (i), (ii) implies the standard version of Jordan-Hoelder theorem (about modules of finite length).