Problem assignment 2.

Algebraic Theory of D-modules.

Joseph Bernstein

(*) **1.** (i) Let k be a field of positive characteristic p. Consider the Weyl algebra $A = A_n$ over k generated by $x_1, ..., x_n, \partial_1, ..., \partial_n$ with the usual commutation relations.

Show that there exists an A-module M which is finite dimensional as a k vector space.

(ii) Show that all simple A-modules are finite dimensional over k.

(iii) Give a classification of simple A-modules in case when the field k is algebraically closed.

Definition. Let V be a vector space.

An (increasing) **filtration** F on V is a collection of subspaces $V_k = F_k V$ for all $k \in \mathbf{Z}$ satisfying the following properties

(i) V_k is an increasing sequence, i.e. $V_i \subset V_k$ for $i \leq k$.
(ii) **Separation.** $V_k = 0$ for $k \ll 0$
(iii) **Exhaustion.** $V = \bigcup V_k$.

A morphism of filtered vector spaces from V, F to W, Ψ is a linear operator $\nu : V \to W$ such that $\nu(V_k) \subset W_k$ for all k.

For a filtered vector space V, F we denote by $gr^F(V)$ the associated **graded** vector space defined as follows:

$gr^F(V) := \oplus_k gr^F_k(V)$, where $gr^F_k(V) := V_k/V_{k-1}$

2. Let V, F be a filtered vector space. Let L be a subspace of V and $N = V/L$ be the quotient space, so we have a short exact sequence $0 \to L \to V \to N \to 0$.

Show that there exist unique pair of filtrations Φ on L and Ψ on N such that morphisms $L \to V$ and $V \to N$ are morphisms of filtered spaces and the corresponding morphisms

$0 \to gr^\Phi L \to gr^F V \to gr^\Psi N \to 0$ form a short exact sequence.

Describe these filtrations explicitly. Characterize them by universal properties.

These filtrations are called **induced filtrations** on subspace and quotient space.

Definition. (i) Let A be an associative algebra with 1. A filtration A_k on A is called an **algebra filtration** if it satisfies

(i) $A_k A_l \subset A_{k+l}$
(ii) $1 \in A_0$

A filtered algebra is an algebra equipped with an algebra filtration.

(ii) Let A be a filtered algebra and M an A-module. A **module filtration** on M is a filtration M_i which satisfies $A_k M_l \subset M_{k+l}$

An A module M equipped with a module filtration we will call a **filtered A-module**.

3. (i) Show that if A, F is a filtered algebra then the space $\Sigma = gr^F A$ has a natural structure of a **graded** associative algebra with 1.

(ii) Show that if M is a filtered A-module then the space $M_\Sigma := grM$ has a natural structure of a **graded** Σ-module.

Definition. Let A, F be a filtered algebra and M, Ψ be a filtered A-module. We say that Ψ is a **good** filtration if it satisfies

(i) Every module M_k is a finitely generated A_0-module.
(ii) For large k $A - 1M_k = M_{k+1}$

4. Show that a (module) filtration Φ on M is a good filtration if and only if the associated graded module $M|_\Sigma = gr^\Phi M$ is a finitely generated Σ-module.

5. Let Σ be a graded algebra. Show that it is Noetherian iff it satisfies the following conditions

(i) Σ_0 is a Noetherian algebra
(ii) For every k the space Σ_k is a finitely generated Σ_0-module.
(iii) Algebra Σ is finitely generated over Σ_0.

[P] 6. Let A, F be a filtered algebra. Let us assume that the associated graded algebra Σ is Noetherian.

(i) Let Φ be a good filtration on an A-module M. Let L be a submodule of M and $N := M/L$. Show that the induced filtrations Φ_L on L and Φ_N on N are good.
(ii) Show that an A-module M admits a good filtration if and only if it is finitely generated.
(iii) Show that in this case the algebra A is Noetherian.
(iv) Show that any two good filtrations are comparable.
Namely, if Φ and Ψ are good filtrations on M then there exists N such that for any k, $\Phi^k \subset \Psi^{k+N}$ and $\Psi^k \subset \Phi^{k+N}$.

7. Let f be an integer sequence. Show that the following are equivalent:
(i) f is eventually polynomial of degree d
(ii) Δf is eventually polynomial of degree $d - 1$
(iii) $f \sim \sum_{i=1}^{d} e_i B_i$, where $e_i \in \mathbf{Z}$ and B_i are polynomial sequences given by $B_i(k) = C_k^i$.

[P] 8. Let $\mathfrak{g}$ be a Lie algebra. Let $U = U(\mathfrak{g})$ be the universal enveloping algebra with the natural filtration.

Let M be a $\mathfrak{g}$ -module, which we can consider as a U-module. Fix a good filtration Φ on M.
Show that the function $k \mapsto dimM_k$ is eventually polynomial.
The degree of this polynomial is called the **Gelfand-Kirillov dimension** of the module M.

[P] 9. Show that in case of a field of characteristic 0 the center of the Weyl algebra consists of scalars.

Describe explicitly the center for the case of a field of characteristic p (see problem 1).

10. Using Kaplansky trick prove the following

Theorem (Nullstellensatz). Let k be an algebraically closed field, $P = k[x_1, ..., x_n]$ the algebra of polynomial functions in n variables over k.

Let $J \subsetneqq P$ be a proper ideal of P. Then there exists a point $a = (a_1, ..., a_n)$ which is a common zero for all functions in J.

Hint. **[P]** Do this in the case of an uncountable field k.

(*) Reduce the case of arbitrary field k to the case of an uncountable field.

[P] 11. Consider the Weyl algebra A in n variables with the standard filtration discussed in class (x_i, ∂_i generate A_1).

Let M, Φ be a filtered A-module. Suppose we know that for some number d and some constant C we have an estimate of the form $\dim M_k \leq Ck^d$ for large k.

Show that if $d < n$ then $M = 0$.

Show that if $d = n$ then the module M is holonomic and hence has finite length. Give a bound for its length.