On the notion of Hilbert polynomial.

Algebraic Geometry and Commutative Algebra

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I. **Preparation about sequences.** Consider the group F consisting of sequences of rational numbers $f = \{f(i), i \in \mathbf{Z}\}$. Let us introduce an equivalence relation on F by $f \sim h$ if $f(i) = h(i)$ for $i \gg 0$.

We say that a sequence f is **eventually polynomial** if there exists a polynomial $P \in \mathbf{Q}[t]$ such that f is equivalent to the sequence $P(i)$. It is clear that such polynomial P is uniquely defined.

Consider the difference operator $\triangle : F \to F$ defined by $\triangle(f)(i) = f(i+1) - f(i)$

1. Let d be a natural number. Show that a sequence $f \in F$ is eventually polynomial of degree $\leq d$ iff $\triangle^{d+1}(f) \sim 0$; this is also equivalent to the condition that $\triangle(f)$ is eventually polynomial of degree $\leq d-1$.

II. **Hilbert polynomial.** Fix an arbitrary field K. Consider an algebra $A = K[x_1, ..., x_n]$ and introduce on it **algebra filtration** $\{A_k\}$, where $A_k = \{P \in A |\ deg\ P \leq k\}$.

Let M be a finitely generated A-module. Fix a system of generators $m_1, ..., m_r$ and consider a filtration of M defined by $M_k = A_k m_1 + A_k m_2 + ... + A_k m_r$.

Our goal is to prove the following fundamental result due to Hilbert.

Theorem A. The sequence $f_M(i) = \dim M_i$ is eventually polynomial (here dimension is over the field K).

It is convenient to formulate and prove slightly more general result.

Definition. (i) A **filtration** of M is a collection of finite dimensional subspaces $M_k \subset M$ defined for all $k \in \mathbf{Z}$ that satisfies the following conditions.

(a) $M_k \subset M_l$ for $k \leq l$, $M_k = 0$ for $k \ll 0$ and $\bigcup M_k = M$.

(b) $A_k M_l \subset M_{k+l}$

(ii) Filtration $\{M_k\}$ is called **good filtration** if it satisfies

(c) For large k we have $A_1\ M_k = M_{k+1}$.

Clearly the filtrations considered in Theorem A are good. So we will prove more general result

Theorem B. Suppose $\{M_k\}$ is a good filtration of an A-module M.

(i) For any A-submodule $L \subset M$ consider the induced filtration on L defined by $L_k = L \bigcap M_k$. Then it is a good filtration.

(ii) The sequence $f(i) := \dim M_i$ is eventually polynomial.

Consider the graded algebra $C = K[t_0.t_1, ..., t_n]$. Using the filtration $\{M_k\}$ on M construct a graded C-module $N = \hat{M} \subset M[t, t^{-1}]$ by $\hat{M}^k = M_k t^k$, where t_0 acts as multiplication by t and t_i acts as a multiplication by tx_i for $i = 1, ..., n$.

2. Check that a filtration $\{M_k\}$ is good iff the C-module $\hat{M}$ is finitely generated.

For an A-submodule $L \subset M$ consider the induced filtration $\{L_k\}$. Then $\hat{L}$ is a D-submodule of D-module $\hat{M}$. Hence Hilbert basis theorem implies (i).

It is clear that the theorem B follows from the following

Theorem C. Consider the algebra $C = K[t_0, t_1, ..., t_n]$ and define the grading $C = \bigoplus C^k$ on it by condition $deg(t_i) = 1$. Fix a graded C-module $N = \bigoplus N^k$.

Suppose we know that C-module N is finitely generated. Then the sequence $f_N(i) := \dim N^i$ is eventually polynomial of degree $\leq n$.

Proof. Consider the operator $T : N \to N$ of degree 1 given by multiplication by t_n. Let us denote by K and C its kernel and cokernel.

3. Check that $f_N(i+1) - f_N(i) \equiv f_C(i+1) - f_K(i)$

Now note that on the modules K and C the operator t_n is zero, so they are finitely generated modules over the algebra $C' = K[t_0, t_1, ..., t_{n-1}]$.

Using induction in n we can assume that the sequences f_K and f_C are eventually polynomial of degree $\leq n-1$. But then it means that the sequence $\triangle(f)$ is eventually polynomial of degree $\leq n-1$ and hence f is eventually polynomial of degree $\leq n$.

Remarks. (i) Note that in fact we start our induction from the case $n = -1$, i.e. $C = K$.

(ii) The most non-trivial step in this proof is the fact that the C-module K is finitely generated - this is Hilbert's basis theorem.

III. **Some problems about Hilbert polynomials.**

4. Let $\mathcal{O}$ be a finitely generated K-algebra and M a finitely generated $\mathcal{O}$-module.

Let us fix a system of generators $x_1, ..., x_n \in \mathcal{O}$. Then M becomes a module over the polynomial algebra $A = K[x_1, .., x_n]$.

Let us choose a good filtration on M and consider the corresponding Hilbert polynomial $f_M(i)$.

(i) Show that the degree $d(M)$ of the polynomial f_M and its first coefficient $e(M)$ do not depend on the choice of a good filtration on M.

(ii) Show that the degree $d(M)$ does not depend on the choice of generators of the algebra $\mathcal{O}$.

We call this invariant $d(M)$ the "the functional dimension" of M.

5. (i) Show that if L is an $\mathcal{O}$-submodule of M then $d(M) = \max(d(L), d(M/L))$.

(ii) Let A be an endomorphism of an $\mathcal{O}$-module M. Show that if A is injective then $d(M/AM)$ is strictly less then $d(M)$ (we assume $M \neq 0$).

(iii) Suppose that we have a vector space M that is a module over two commutative finitely generated algebras A and B. Let us assume that it is finitely generated over A and also over B, so we can define two invariants $d_A(M)$ and $d_B(M)$.

Show that if the actions of A and B on the module M commute, then $d_A(M) = d_B(M)$.

6. Let X be an affine algebraic variety, M a finitely generated $\mathcal{O}(X)$ -module. We define the support of M to be the subset $\sup(M) \subset X$ defined by the ideal $I = Ann(M) \subset \mathcal{O}(X)$.

Show that $d(M)$ equals $\dim \sup(M)$.

7. Prove that the dimension function $\dim_H(X)$ defined using Hilbert polynomial definition has the following properties. Let $\pi : X \to Y$ be a morphism of affine algebraic varieties

(i) Suppose that π is a finite morphism (e.g. a closed embedding). Then $\dim_H X \leq \dim_H Y$.

(ii) Suppose that π is a finite epimorphism. Then $\dim_H X = \dim_H Y$.

(iii) Suppose π is an imbedding of a basic open subset (i.e. $X = Y_f$). Then $\dim_H X \leq \dim_H Y$

8. Show that Hilbert polynomial definition of dimension for algebraic varieties is equivalent to Krull's definition.

(*) **9.** Using Hilbert polynomial definition of dimension prove directly the **Principle ideal theorem.**

Let X be an irreducible affine algebraic variety, $f \in \mathcal{O}(X)$, $Z = Z(f)$ the zero set of the function f. Suppose that $\dim Z \leq \dim X - 2$. Then Z is empty.