Problem assignment 12.

Algebraic Geometry and Commutative Algebra

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1. Let X be a topological space and $0 \to L \to M \to N \to 0$ a s.e.s (short exact sequence) of sheaves on X. Let us fix a section $\xi \in N(X)$; we would like to find its lifting to M, i.e. a section $\delta \in M(X)$ such that $p(\delta) = \xi$ (here p denotes the morphism $p : M \to N$ above).

Suppose X is a union of two open subsets U and W. Suppose we found sections δ_U and δ_W over these subsets.

(i) Show that if L is flabby then there exists a section $\delta \in M(X)$ such that $p(\delta) = \xi$ and $\delta|_U = \delta_U$

(ii) Show that in general if the sheaf L is flabby then the morphism $p : M(X) \to N(X)$ is onto.

Hint. Use Zorn's lemma.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an additive functor (you may always assume that these categories are some categories of modules).

2. (i) Show that the functor F is exact iff it maps s.e.s into s.e.s.

(ii) We say that the functor F is **left exact** if it maps any left short exact sequence (l.s.e.s) $0 \to L \to M \to N$ into a l.s.e.s.

Show that F is left exact iff it maps any s.e.s into l.s.e.s.

Some cohomological constructions.

Consider the category of complexes $Com(\mathcal{A})$. Usually we denote complex as $C^{\cdot}$ meaning $... \to C^i \to C^{i+1} \to ....$

For every complex $C^{\cdot}$ we denote by $H^i(C^{\cdot})$ its cohomology groups. We say the $C^{\cdot}$ is acyclic (= exact) at place i if $H^i(C^{\cdot}) = 0$. Similarly for any collection of places i.

Check that $Com(\mathcal{A})$ is an abelian category.

[P] 3. (i) Show that a morphism of complexes $\nu : C^{\cdot} \to D^{\cdot}$ induces morphisms of cohomology groups $\nu_* : H^i(C) \to H^i(D)$.

(ii) Let $0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$ be a short exact sequence of complexes.

Construct connecting morphisms $\delta^i : H^i(C) \to H^{i+1}(A)$ and show that the long sequence of cohomologies is exact.

Show that the construction of the connecting morphisms is functorial.

Definition. A morphism of complexes $\nu : C^{\cdot} \to D^{\cdot}$ is called **quasiisomorphism** if it induces an isomorphism on cohomologies.

4. Prove **Five lemma.** Let $L^{\cdot}, M^{\cdot}$ be two complexes and $\nu : L^{\cdot} \to M^{\cdot}$ be a morphism of complexes.

Let us assume that the complexes are exact, morphisms ν_1 and ν_{-1} are isomorphisms, ν_{-2} is epimorphic and ν_2 is mono.

Sow that in this case the morphisIn ν_0 is an isomorphism.

Definition. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. A collection of objects $Q \subset ISO(\mathcal{A})$ we call **adapted to** F if it satisfies the following conditions

(ad1) for any s.e.s $0 \to L \to M \to N \to 0$ we have
(i) If $L, M \in Q$ then $N \in Q$
(ii) If $L \in Q$ then $0 \to F(L) \to F(M) \to F(N) \to 0$ is a s.e.s.

(ad2) The family Q is rich, i.e. every object can be imbedded into an object $D \in Q$.

[P] 5. Show that using the family Q adapted to F we can construct the right derived cohomological functor $RF = \{R^iF\}$.

Hint. Show that any object M has a right Q-resolution R. Show that for any two Q resolutions R, R' of M there exists a Q resolution R'' that contains R and R'. Show that in this case the imbedding $R \to R''$ induces a quasiisomorphism $F(R) \to F(R'')$ (and similarly for R').

Show that for any morphism $\nu : M \to N$ and any Q resolution R of M we can find a Q resolution P of N and extend ν to a morphism of resolutions $\nu' : R \to P$.

Definition. **Cone construction.** Let $\nu : L \to M$ be a morphism of complexes. We construct a new complex $Cone(\nu)$ as follows. We extend ν to a complex of complexes placing $L^\cdot$ and $M^\cdot$ in places -1 and 0, consider the corresponding bicomplex B and set $Cone(\nu) := Tot(B)$.

[P] 6. (i) Write explicit formulas for the complex $Cone(\nu)$. Show that there exists a short exact sequence of complexes $0 \to M \to Cone(\nu) \to L[1] \to 0$.

Deduce from this a long exact sequence connecting cohomologies of L, M and $Cone(\nu)$.

(ii) Show that the morphism of complexes ν is a quasiisomorphism iff the complex $Cone(\nu)$ is acyclic.

(ii) Show that if ν is injective then $Cone(\nu)$ is quasiisomorphic to the quotient complex M/L.

7. Let $\nu : B \to B'$ be a morphism of bicomplexes. Suppose we know that for every row ν induces quasiisomorphism of the complexes $\nu : Row^j(B) \to Row^j(B')$. Show (under appropriate finiteness assumptions) that ν induces a quasiisomorphism of total complexes $Tot(B) \to Tot(B')$.

(**Hint.** Using problem 6 (ii) reduce this statement to Grothendieck's lemma).

Truncation. Let k be an integer. We define the lower truncation functor $\tau_{\leq k}$ from category of complexes into itself as follows. For a complex $M^\cdot$ we consider subcomplex $L =: \tau_{\leq j} M$, where $L^i = M^i$ for $i < k$, $L^i = 0$ for $i > k$ and $L^k = \ker(M^k \to M^{k+1})$.

8. Show that $H^i(L) = H^i(M)$ for $i \leq k$ and $H^i(L) = 0$ for $i > k$.

Compute cohomologies of the quotient complex M/L.

[P] 9. Let B^{ij} be a bicomplex. Suppose that every row is acyclic outside column 0. Prove (under appropriate finiteness assumptions) that the total complex $Tot(B)$ has the same cohomologies as the complex combined from objects $L^i := H^0(Row_i(B))$.

In fact these complexes are quasiisomorphic.