Problem assignment 2.

Algebraic Geometry and Commutative Algebra

Joseph Bernstein November 12, 2008.

A remark on problems in different areas. In my assignments I will try to single out problems that are not directly related to algebraic geometry. For example sign (CA) signifies a problem (or a definition) from commutative algebra, (LA) stands for linear algebra.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Dmitry).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

Remark. In this assignment you can freely use the Nullstellensatz.

1. (T) Let X be a topological space, $Y \subset X$. Show that the following conditions are equivalent:

(i) Y is an intersection of an open and a closed subsets of X.

(ii) Y is locally closed, i.e. any point $x \in Y$ has an open neighborhood U such that $U \cap Y$ is closed in U .

(iii) Y is open in $cl(Y)$ (here $cl(A)$ is the closure of A).

2. Let X be an algebraic variety and $Z \subset X$ a locally closed subset. Show that Z has a canonical structure of an algebraic variety.

Here you should use the following finiteness lemma that we will prove soon

Lemma. Let X be an affine algebraic variety and U an open subset of X. Then U can be covered by a finite number of basic open subsets X_f.

3. Let X be an algebraic variety. Fix an open covering of X consisting of affine algebraic varieties B_i.

Show that a function ϕ on X is regular iff its restriction to every subset B_i is polynomial. Show that we can express the space of regular functions $\mathcal{O}(X)$ as the kernel of a morphism $\nu : \oplus_i \mathcal{P}(B_i) \to \oplus_{i,j} \mathcal{P}(B_i \bigcap B_j)$. Here we assume that all the intersections $B_i \bigcap B_J$ are affine - this will be true for most of interesting cases.

4. Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a space with functions. Let $p : X \to Y$ be an epimorphic map of sets.

Show that in this case we can canonically define on Y the structure of a space with functions. Work this out in two cases:

(i) Let V be a finite dimensional k-vector space. Consider the algebraic variety $X = \mathbf{V}^* := \mathbf{V} \setminus 0$ and the natural projection of sets $p : X = \mathbf{V}^* \to Y = P(V)$.

(ii) Let $V = k^2$. Consider the algebraic variety $X = \mathbf{V}^*$. Define the action of the group $H = k^*$ on V^* by $a(x, y) = (ax, a^{-1}y)$ and consider the quotient set $Y = X/H$.

Show that in both cases we get on Y the structure of an algebraic variety. In first case we get the projective space $\mathbf{P}(V)$, in the second case we get the standard example of non-separate algebraic variety (we will discuss this later).

[P] 5. Let V be a vector space as above and $\mathbf{P} = \mathbf{P}(V)$ the corresponding projective space. We would like to describe it more explicitly.

Denote by A the algebra of polynomial functions on V. This is a graded algebra $A = \oplus A^k$.

Given a homogeneous polynomial $f \in A^k$ with $k > 0$ we denote by V_f the corresponding basic open subset in $\mathbf{V}$ and by $\mathbf{P}_f$ its image in $\mathbf{P}$.

(i) Show that the sets $\mathbf{P}_f$ form a basis of the Zariski topology on $\mathbf{P}$.

(ii) Show that every subset $\mathbf{P}_f$ is an affine algebraic variety and the algebra of regular functions $\mathcal{O}(\mathbf{P}_f)$ ia isomorphic to the subalgebra A_f^0 of functions of degree 0 in the graded algebra A_f.

In order to do this you will need a result from linear algebra described in the next problem.

Definition. (LA) Let H be a group. By definition a **character** of H is a homomorphism $\chi : H \to k^*$.

Suppose we fixed an action of the group H on a k-vector space L. For any character χ of H we consider eigen subspace

$L^\chi := \{v \in L | hv = \chi(h)v \text{ for all } h \in H\}$

[P] 6. (LA). Let a group H act on a space L. Fix a collection of characters $\chi_1, ..., \chi_m$ of H pairwise distinct.

Suppose we have an equality in L of the form $v = \sum_i v_i$, where $v_i \in L^{\chi_i}$. Show that every vector v_i can be written as a linear combination of vectors hv for some elements $h \in H$.

In particular show that if $v = 0$ then all vectors v_i are 0.

[P] 7. Let A be a finitely generated k-algebra. Set $M(A) := \mathrm{Mor}_{k-alg}(A, k)$.

(i) Describe Zariski topology on the set $M(A)$. Show that the set $M(A)$ has a natural structure of an affine algebraic variety. In particular describe the algebra of regular functions $\mathcal{O}(M(A))$.

(ii) Given a morphism of k-algebras $\mu : B \to A$ show that the corresponding map $\nu : M(A) \to M(B)$ is a morphism of affine algebraic varieties.

Describe the closure of the image of ν. Give an example when the morphism ν has finite fibers but is not closed.