Problem assignment 3.

Algebraic Geometry and Commutative Algebra

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1. (LA) Let V be a vector space over an algebraically closed field k, $a : V \to V$ a linear operator.

(i) Suppose a is locally nilpotent, i.e. for any vector $v \in V$ we have $a^n v = 0$ for large n. Show that then $Spec(a) = \{0\}$ (i.e. for every $\lambda \neq 0 \in k$ operator $a - \lambda$ is invertible)

(ii) Show that if the space V is countable dimensional and the field k is algebraically closed and uncountable, then conversely the condition $Spec(a) = \{0\}$ implies that a is locally nilpotent.

2. (CA). Let A be a ring and M an A-module.

(i) Show that M is finitely generated iff it satisfies the following condition:

(*) Let $M_\alpha \subset M$ be a directed system of submodules such that the union $\bigcup M_\alpha$ equals M. Then it contains M.

(ii) Show that M is Noetherian iff it satisfies the following condition:

(**) Any directed system of submodules $M_\alpha \subset M$ has a maximal element.

[P] 3. (i) Let $A = k[t_1, t_2, ...]$ be the algebra of polynomials in infinite number of generators. Show that A is not Noetherian

(ii) Let $A = k[t_1, t_2]$. Find an example of a k-subalgebra $B \subset A$ which contains 1 such that B is not Noetherian (and hence not finitely generated as k-algebra).

4. (i) Consider homomorphisms of algebras $C \to B \to A$. Show that if A is finite over B and B is finite over C then A is finite over C.

(ii) Choose a monic polynomial $P \in B[t]$ and consider the B-algebra $A = B[t]/PB[t]$. Show that A is finite over B.

5. Here is an elementary proof of Nakayama lemma.

Jet J be an ideal in a commutative ring A. We set $R = 1 + J \subset A$. Clearly $R \cdot R \subset R$ and $R + J \subset R$.

Lemma (Nakayama). Let M be a finitely generated A-module such that $JM = M$. Then there exists an element $r \in R$ such that $rM = 0$.

In particular, for any submodule $L \subset M$ we have $JL = L$.

Induction in number n of generators. Let x be one of generators of M and $N = Ax \subset M$ the submodule generated by x.

Using the induction assumption for the module M/N we can find an element $r_1 \in R$ such that $r_1 M \subset N$.

This implies that $r_1 x \in r_1 JM = J r_1 M \subset JN = Jx$.

But this shows that there exists an element $r_2 \in R$ such that $r_2 x = 0$ and hence $r_2 N = 0$.

Thus for $r = r_1 r_2$ we have $rM = 0$.

6. Proof of Hamilton - Cayley identity.

Lemma Let $R \in Mat(n, C)$ be a $n \times n$ matrix over a commutative ring C. Then there exists an **adjoint** matrix $Q \in Mat(n, C)$ such that $QR = det(R) \cdot 1_n$.

Now let A be a commutative ring, $S \in Mat(n, A)$. Set $C = A[t]$ and define the characteristic polynomial $P \in C$ of the matrix S to be $det(R)$, where $R = t1_n - S \in Mat(n, C)$.

Theorem (Hamilton-Cayley) $P(S) = 0$.

For the proof consider the action of the algebra $Mat(n, C) \simeq Mat(n, A)[t]$ on the group $H = Mat(n, A)$ where the subalgebra $Mat(n, A) \subset Mat(n, C)$ acts on H by left multiplication and the element t acts as right multiplication by the matrix S.

Let $h \in H$ be the identity matrix. It is clear that $R(h) = 0$. This implies that for adjoint matrix Q of R we have $QR(h) = 0$, i. e. $P(t)(h) = 0$. But it is clear that $P(t)(h) = P(S)$ and thus $P(S) = 0$.

[P] 7. Let M be a finitely generated module over a commutative ring C.

(i) Let X be an endomorphism of M. Show that there exists a monic polynomial $P \in C[t]$ such that $P(X) = 0$.

(ii) Let A be a commutative finitely generated C-subalgebra of $End_C(M)$. Show that it is finite over C.

(iii) Let J be an ideal of C. Suppose that the operator X in question (i) satisfies $XM \subset JM$.

Show that then we can choose the monic polynomial $P \in C[t]$ in question (i) of the form $P = \sum a_i t^{n-i} | i = 0, 1, ..., n$ in such a way that $a_0 = 1$ and $a_i \in J^i$ for all i.

[P] 8. Let A be any ring. Consider full subcategory $No(A) \subset \mathcal{M}(A)$ of Noetherian A-modules

(i) Show that this subcategory is closed with respect to subquotients and extensions.

(ii) Consider the algebra $D = A[t]$; for every A-module M define D-module $M[t]$.

Show that if M is a Noetherian A-module then $M[t]$ is a Noetherian D-module.

[P] 9. (LA) Let L be a finite dimensional vector space over an algebraically closed field k. Let A be a commutative subalgebra in $End(L)$.

(i) Show that if $L \neq 0$ then there exists a non-zero common eigenvector $v \in L$ such that $av = \chi(a)v$ for some character χ; we call the vector v eigenvector and the character χ the corresponding eigencharacter.

(ii) Let $v_1, , v_m$ be eigenvectors corresponding to characters χ_i. Assume that the characters χ_i are pairwise distinct. Show that then the vectors v_i are linearly independent. In particular, if they are non-zero, we have $m \leq \dim L$.