Problem assignment 4.

Algebraic Geometry and Commutative Algebra

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Definition. Let X be a topological space. The space X is called **irreducible** if for any two proper closed subsets $F, F' \subsetneqq X$ we have $F \bigcup F' \neq X$. A subset $Z \subset X$ is called **irreducible** if it is irreducible in induced topology.

[P] 1. (i) Show that X is irreducible iff it satisfies the following condition:

(*) Every non-empty open subset $U \subset X$ is dense in X.

(ii) Show that a subset $Z \subset X$ is irreducible iff its closure $cl(Z)$ is irreducible.

(iii) Let $\nu : X \to Y$ be a continuous map of topological spaces. Show that if a subset $Z \subset X$ is irreducible then its image $\nu(Z)$ is irreducible.

Definition. Let X be a topological space. A subset $Z \subset X$ is called **constructible** if it is a union of finite number of locally closed subsets of X.

2. (i) Show that constructible sets form an algebra (i.e. the family of constructible sets is closed with respect to finite unions and intersections and with respect to taking the complement).

(ii) Let $Z \subset X$ be a constructible subset. Show that Z contains a subset $U \subset Z$ which is open and dense in the closure $cl(Z)$.

[P] 3. (i) Let Y be a locally closed subset of an algebraic variety X. Denote by $cl(Y)$ the closure of Y in X and define the **boundary** ∂Y of Y by $\partial Y = cl(Y) \setminus Y$.
Show that $\dim cl(Y) = \dim Y$ and $\dim \partial Y < \dim Y$.

(ii) Let $\mathcal{S} = (S_1, ..., S_k)$ be any stratification of an algebraic variety X (i.e. its decomposition as a disjoint union of locally closed subvarieties). Show that $\dim X = \max \dim S_i$.

(ii) Let $X_1, ..., X_m$ be irreducible components of X. Show that $\dim X = \max \dim X_i$.

Definition. If X is an algebraic variety and $a \in X$ we define the local dimension $\dim_a X$ to be the minimum of $\dim U$ for all open neighborhoods U of a.

[P] 4. Suppose X is irreducible. Show that for any point $a \in X$ the local dimension $\dim_a X$ equals $\dim X$.

[P] 5. Let X be an affine irreducible variety. Denote by $k(X)$ the field of rational functions on X. Show that the dimension $\dim X$ equals to the transcendence degree of the field $k(X)$ over k.

6. Let X be an algebraic variety. Consider a function $Z \mapsto \dim Z$ defined on locally closed subsets of X.
Show that this function can be extended to all constructible subsets $Z \subset X$ so that if $Z = \coprod Z_i$ (finite disjoint union), then $\dim Z = \max \dim Z_i$. Show that this extension is uniquely defined.

In problems bellow you can use the strong form of Chevalley's Theorem.

[P] 7. Let $\nu : X \to Y$ be a morphism of algebraic varieties.
Show that if ν is **dominant**, i.e. its image is dense in Y, then $\dim X \geq \dim Y$.
Show that if ν has finite fibers then $\dim X \leq \dim Y$.

Definition. A function u on a topological space X is called **constructible** if it takes finite number of values and every level set of it constructible.

The general ideology of algebraic geometry is that if u is a function on an algebraic variety X with integer values which is "algebraically defined", then it is always constructible.

Let X be a variety over a base S, i.e. it is given together with a morphism $p_X : X \to S$ of algebraic varieties. We interpret S as a base and consider the family of varieties $X_s := (p_X)^{-1}(s)$ for $s \in S$ as an "algebraic family of varieties" parameterized by points of S.

Similarly, given two varieties X, Y over S and a morphism $\nu : X \to Y$ over S (formulate precisely what it means) we get an "algebraic family of morphisms $\nu_s : X_s \to Y_s$ parameterized by points of the base S.

We would like to consider natural functions and properties depending on points of $s \in S$ which describe some algebraic properties of varieties X_s and morphisms ν_s.

[P] 8. (i) Consider the function $u(s) := \dim X_s$. Show that u is a constructible function on S.

(ii) Suppose we know that all varieties X_s have dimension 0. Consider the function $h(s) := \#(X_s)$.
Show that the function h is bounded.

(*) (*) (iii) Show that the function h in (ii) is constructible.

[P] (*) **9.** Consider the situation in problem 8 and assume for simplicity that S, X, Y are affine. Given some property $\mathbf{P}$ of morphisms of algebraic varieties let us consider the subset $S_{\mathbf{P}} \subset S$ consisting of points s such that the morphism ν_s satisfies $\mathbf{P}$.

Show that the subsets $S_{\mathbf{P}}$ are constructible for the following properties:

(i) imbedding (ii) closed imbedding
(iii) epimorphism (iv) dominant morphism
(v) finite morphism (vi) morphism with finite fibers

(*) (*) (*) **9.** In problem 8 Consider the function c on the base S defined as follows $c(s) :=$ number of irredicible components of X_s.
Show that this function is constructible.

[P] ∇ **10.** (CA) Let A be a ring (commutative with 1). Denote by $Spec(A)$ the set of its prime ideals.

(i) Introduce the Zariski topology on $Spec(A)$.

(ii) Show that the intersection of all prime ideals equals to the Nil radical of A.

(iii) Suppose we know that the ring A is Noetherian.
Show that A has a finite number of minimal prime ideals and that every prime ideal of A contains some minimal prime ideal. In particular, show that the intersection of the minimal prime ideals of A equals to the Nil radical of A.