Problem assignment 5.

Algebraic Geometry and Commutative Algebra

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Some problems about finite algebras (CA).

Rings that we consider are commutative with 1; morphisms of rings are assumed to preserve 1.

Let C be a ring. By definition **a C-algebra** is a ring A together with a specified morphism of rings $\nu : C \to A$. In particular, A is a C-module.

Definition. We say that a C-algebra A is **finite over** C if it is finitely generated as C-module. Note that this is equivalent to the condition that A is finite over the subalgebra $C' = \nu(C) \subset A$.

1. Consider morphisms of rings $C \to B \to A$. Show that if A is finite over B and B is finite over C then A is finite over C.

Definition. Let A be a C-algebra. An element $a \in A$ is called **integral** over C if there exists a monic polynomial $P \in C[t]$ such that $P(a) = 0$.

[P] 2. Show that the following conditions on an element $a \in A$ are equivalent

(a) a is integral over C.

(b) The subalgebra $C < a > \subset A$ is finite over C.

(c) There exists a subalgebra $B \subset A$ that contains $C < a >$ and is finite over C.

[P] 3. Let A be a finitely generated C-algebra. Show that the following conditions are equivalent

(a) A is finite over C,

(b) Every element $a \in A$ is integral over C.

(c) There exists a finite system of generators $x_1, ..., x_m$ of A over C which are all integral over C.

[P] 4. Let X be an algebraic variety and $Z \subset X$ its closed subset. Suppose we know that one of irreducible components T of the variety Z has dimension m. Sow that there exists an open affine subset $U \subset X$ such that $Z \bigcap U$ is an irreducible closed subset of U of dimension m.

Some problems about UFD (unique factorization domains).

∇ **5.** Let A be a unique factorization domain, L its field of fractions. Consider subring $B = A[t] \subset L[t]$.

(i) Prove **Gauss lemma.** Let $P, Q \in L[t]$ be monic polynomials. Suppose that $R = PQ$ lies in B. Show that then P and Q also lie in B.

(ii) Using (i) show that for any field K the algebra $K[x_1, ...x_n]$ is a unique factorization domain.

[P] 6. Let X be an irreducible algebraic variety of dimension n. Let us denote by H the set of all closed irreducible subvarieties $H \subset X$ of dimension $n-1$. We define the group of divisors $Div(X)$ as a free abelian group generated by H (this group consists of linear combinations $\sum_H a_H H$ where $a_H \in \mathbf{Z}$ and all a_H except finite number are 0).

Suppose X is affine and the algebra $A = \mathcal{P}(X)$ is UFD. Denote by L the field of fractions of A.

Show that we have a natural isomorphism $Div(X) = L^*/A^*$.

[P] 7. Consider subvariety $X = V(xy - z^2) \subset \mathbf{A}^3$.

(i) Prove that the y-axis L is a subvariety of X of codimension 1, but the ideal $J(L) \subset \mathcal{O}(X)$ is not principal. Show that some power of this ideal is principal.

(ii) Show that $\mathcal{O}(X)$ is not a unique factorization domain.

Definition. Let Y be an irreducible algebraic variety, P a property which holds for some points $y \in Y$. We say that the property P holds for **generic point** of Y if the set of points for which P holds contains an open dense subset of Y.

[P] 8. Let $\pi : X \to Y$ be a dominant morphism of irreducible algebraic varieties of relative dimension k (i.e. $k = \dim X - \dim Y$). For every point $y \in Y$ consider the fiber $F_y = \pi^{-1}(y)$.

(i) Show that for generic point $y \in Y$ $\dim F_y = k$.

(ii) Show that for every point $y \in Y$ dimension of every irreducible component of the fiber F_y is $\geq k$.

9. Let V be a finite dimensional vector space over k and $\mathbf{V}$ the corresponding affine variety.

(i) Fix a number l. Define the structure of an algebraic variety on the set G_l of all affine (i.e not necessarily passing through 0) linear subspaces $L \subset V$ of codimension l.

(ii) Prove the following

Proposition. Let Y be an algebraic subvariety of $\mathbf{V}$. Show that the following conditions are equivalent:

(a) $\dim Y \leq k$

(b) For generic point $L \in G_l$ with $l > k$ the space L does not intersect Y.

(c) For generic point $L \in G_k$ the intersection of L with Y is finite.

(**Hint.** Consider the incidence variety $Z \subset Y \times G_l$ consisting of points (y, L) such that $y \in L$ and compute its dimension using projections to Y and to G_l).

This proposition can be used as a definition of dimension, and as a powerful tool for computing dimension of different varieties.