Problem assignment 6.

Algebraic Geometry and Commutative Algebra

Joseph Bernstein December 10, 2008.

Definition. (CAT). Let D be a category. An object $I \in Ob(D)$ is called an **initial object** if for any object $X \in Ob(D)$ the set $Mor_D(I, X)$ has exactly one element. Similarly, a **final** object is an object F such that for any X the set $Mor_D(X, F)$ has exactly one element.

1.. (i) Show that if initial object exists then there might be many of them but they are is defined uniquely up to unique isomorphism. Similarly for final objects.

(ii) Exhibit initial and final objects in the standard categories $Sets, Top$, $Groups, Ab, Modules$.

Definition. Let $\mathcal{A}$ be a category, X, Y objects of $\mathcal{A}$. Let us consider a new category $D = D(X, Y)$ defined as follows: an objet of D is a triple (P, α, β), where $\alpha \in Mor(P, X), \beta \in Mor(P, Y)$. Morphism of the object (P, α, β) to (Q, γ, δ) is a morphism $\nu : P \to Q$ which makes diagrams commutative $\gamma\nu = \alpha, \delta\nu = \beta$.

Suppose the category $D(X, Y)$ has a final object (R, p_X, p_y). Then this object is defined up to unique isomorphism. Corresponding object $R \in Ob(\mathcal{A})$ is called the product of X and Y; standard notation is $X \times Y$.

2. (i) Show that the object $R = X \times Y$ can be uniquely characterized by the property that there exists a functorial bijection of sets $Mor(Z, X) \times Mor(Z, Y) = Mor(Z, R)$ for all $Z \in Ob(\mathcal{A})$.

(ii) Suppose we are given morphisms $\nu : X \to S$ and $\mu : Y \to S$ in the category $\mathcal{A}$ (we can think about X, Y as objects over a base S).

Give a definition and analyze the properties of **fibered product** $X \times_S Y$.

3. (i) Let X, Y be sets. Consider the natural morphism of algebras $F(X) \otimes_k F(Y) \to F(X \times Y)$ (here $F(X)$ is the algebra of all k-valued functions on X). Show that it is injective.

(ii) Let A, B be two commutative finitely generated algebras over a field K. Suppose they do not have nilpotent elements (except 0). Show that if the field K is algebraically closed then the algebra $C = A \otimes_K B$ also has this property.

(iii) Show that (ii) holds without assumption that A and B are finitely generated. Give an example which shows that if the field K is not algebraically closed then the algebra C can have nilpotent elements.

4. (i) Let $p : Z \to X$ be an open continuous map of topological spaces. Show that if X is irreducible and all fibers are irreducible (and hence not empty) then Z is irreducible.

(ii) Show that the product of irreducible algebraic varieties is irreducible. Compute the dimension of the product.

(iii) Let A, B, C be K-algebras in problem 3. Suppose we know that A and B are domains. Show that if the field K is algebraically closed then the algebra C is also a domain.

5. (i) Let X be an algebraic variety. Show that the diagonal morphism $\Delta : X \to X \times X$ defines an isomorphism of X with algebraic subvariety corresponding to a locally closed subset $\Delta(X) \subset X \times X$.

(ii) Show that X is separated iff it satisfies the following condition:

(*) for any open affine subsets $U, V \subset X$ their intersection $W = U \bigcap V$ is also affine and moreover the algebra $\mathcal{O}(W)$ is generated by algebras $\mathcal{O}(U)$ and $\mathcal{O}(V)$.

(iii) Show that a subvariety of a separated variety is separated; show that the product of separated varieties is separated.

(iv) Consider an algebraic variety $A = k^2 \setminus 0$ with an action of the group k^* given by $a(x, y) = (ax, a^{-1}y)$. Let X be defined as A/k^*. Show that X is an algebraic variety. Show that it is not separated.

6. Let us consider the case $k = \mathbf{C}$.

(i) Explain how to introduce on any algebraic variety X the analytic topology $\mathcal{T}_{an}$. We denote the corresponding topological space X_{an}.

Show that $(X \times Y)_{an} = X_{an} \times Y_{an}$. Show that X is separated iff X_{an} is Hausdorff.

7. Let K be a Hausdorff topological space.

(i) Show that if K is compact then it satisfies the following property:

(*) for any Hausdorff topological space S the projection $p_S : K \times S \to S$ is a closed map.

∇ (*) (ii) Show that conversely (*) implies that the space K is compact.

8. Let X, Y be algebraic varieties and Y is separated.

(i) Show that for any morphism $\pi : X \to Y$ its graph $graph(\pi) \subset X \times Y$ is a closed subvariety.

Fix a dense open subset $U \subset X$. We would like to understand to what degree a morphism $\pi : X \to Y$ is determined by its restriction to U.

(ii) Suppose we found another morphism $\tau : X \to Y$ such that $\pi|_U = \tau|_U$. Show that then $\pi = \tau$.

(iii) Suppose we fixed a morphism $\pi : U \to Y$. Show that there exists the maximal open subset $W \supset U$ to which the morphism π can be extended.