Problem assignment 7.

Algebraic Geometry and Commutative Algebra

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1. (CA) Let A be a commutative ring with 1. Fix an element $f \in A$ and consider the localized ring $B = A_f$.

Show that the localization functor $Loc : \mathcal{M}(A) \to \mathcal{M}(B)$ defined by $M \mapsto M_f := M[t]/(1 - tf)M[t]$ is an exact functor and it commutes with arbitrary direct sums.

Show by example that the localization functor does not commute with infinite direct products.

[P] 2. (CA) Let A be a ring as in problem 1. Consider a subset $S \subset A$ (in what follows we can always assume S to be multiplicatively closed which means that $1 \in S$ and if $s, t \in S$ then $st \in S$).

An A-module R is called S-inverting if for every element $s \in S$ the induced endomorphism of the module R given by $r \mapsto sr$ is a bijection.

(i) Fix an A-module M and consider its morphisms into S-inverting modules R. Show that there exists a universal morphism $M \to R_0$ with this property.

In more detail, consider the category D consisting of pairs (R, ν) where R is an S-inverting module and $\nu : M \to R$ a morphism of A-modules. Define what are morphisms in this category and show that this category has an initial object. This object (defined uniquely up to unique isomorphism) we denote (M_S, i).

The module M_S is defined uniquely up to canonical isomorphism. It is called the **localization** of the module M with respect to the subset S.

(ii) Show that the localization functor $M \mapsto M_S$ is exact and commutes with arbitrary direct sums.

(iii) Show that A_S is in fact an algebra, $i : A \to A_S$ a morphism of algebras; identify S-inverting A-modules with A_S-modules.

[P] 3. Let $\nu : F \to G$ be a morphism of sheaves on a topological space X.

(i) Show that ν is epimorphic in the category of presheaves iff for any open subset U the map $\nu : F(U) \to G(U)$ is epimorphic.

Show that ν is an epimorphism in the category of sheaves iff for any point $x \in X$ the map of stalks $\nu : F_x \to G_x$ is epimorphic.

Give an example when ν is an epimorphism of sheaves but not an epimorphism of presheaves.

(ii) Is it possible for ν to be an epimorphism of presheaves, but not an epimorphism of sheaves. The same question about monomorphism and isomorphism.

(iii) Let F be a sheaf of abelian groups and assume that the stalk F_a equals 0 for some point $a \in X$. Does this mean that F vanishes in some neighborhood of a ?

4. Let X be a topological space. Let us consider a system of sets $\mathcal{S} = \{S_x\}$ parameterized by points $x \in X$. A correspondence $\xi : x \mapsto \xi_x \in S_x$ we will call a **section** of the system $\mathcal{S}$ on X.

(i) Define presheaf D by $D(U) := \{$all sections of the system $\mathcal{S}$ on $U\}$. Show that this is a sheaf.

(ii) Suppose that for every open subset $U \subset X$ we have chosen some family of sections $\mathcal{O}(U) \subset D(U)$ (we will call them regular sections). Describe the conditions on these families that ensure that $\mathcal{O}$ is a sheaf on X.

[P] 5. (i) Using construction in problem 4 give a definition of a sheaf F on a topological space X in terms of a collection of stalks F_x - without using the notion of presheaf.

(ii) Let F be a sheaf on X. Show that there exists a unique topology on the set $Y = \coprod F_x$ such that for any open subset U we have $F(U) = \{$set of all continuous sections $\xi : U \to Y\}$.

Use this to give one more definition of a sheaf (original Leray's definition).

6. Let X be an affine algebraic variety, $\mathcal{O}_X$ the structure sheaf on X, $A = \Gamma(X, \mathcal{O}_X)$ the algebra of regular functions on X.

Show that the localization functor defines an equivalence of categories $\mathcal{M}(A) \to \mathcal{M}(\mathcal{O}_X)$. Show that $\mathcal{M}(\mathcal{O}_X)$ is a full subcategory of $Sh(X, \mathcal{O}_X)$. Show that it is generated by free modules by taking cokernels (this means that any object $\mathcal{F} \in \mathcal{M}(\mathcal{O}_X)$ is a quotient of a morphism of free modules).

Geometric meaning of localization functor.

Let M be an A-module and $F = Loc(M)$ the corresponding sheaf. Explain how different geometric objects constructed from F and geometric properties of F can be described in terms of M.

[P] 7. (i) Describe $\Gamma(X, F)$ (ii) For an open subset $U \subset X$ describe $\Gamma(U, F)$

(iii) Describe the fiber $F|_x$ at some point $x \in X$

(iv) Describe the stalk F_x of F at the point x

(v) Describe the property that F is coherent.

(vi) Describe the support $supp(F) \subset X$ (do this first in coherent case).

(vii) Describe the property that F is coherent locally free.

[P] 8. Let X be an algebraic variety.

(i) Show that every quasi-coherent sheaf F of $\mathcal{O}_X$-modules is a union of coherent sheaves of $\mathcal{O}_X$-modules.

(ii) Let F be a coherent $\mathcal{O}_X$-module and $F_1 \subset F_2 \subset ... \subset F_n \subset ...$ an increasing system of submodules. Show that it is stable.

[P] 9. (i) Let F be a coherent $\mathcal{O}_X$-module. Suppose that at some point $a \in X$ the fiber $F|_a$ is 0. Show that there exists a neighborhood U of the point a such that $F|_U = 0$.

(ii) Let $\nu : G \to F$ be a morphism of coherent $\mathcal{O}$-modules on X. Suppose that for some point $a \in X$ the morphism of fibers $\nu : G|_a \to F|_a$ is epimorphic. Show that then ν is epimorphic in some neighborhood U of the point a.

(iii) Is a statement analogous to (ii) correct for monomorphisms ? For isomorphisms ?