Problem assignment 9.

Algebraic Geometry and Commutative Algebra

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I. **Action of groups on $\mathcal{O}$-modules.**

Let us call **$\mathcal{O}$-pair** a pair (X, F) where X is an algebraic variety and F an $\mathcal{O}_X$-module. An isomorphism $\nu : (X, F) \to (Y, H)$ is a pair consisting from an isomorphism $\nu_X : X \to Y$ of algebraic varieties and an isomorphism $\nu' : F \to \nu^*(H)$.

1. Check that these morphisms can be composed and that there exist inverse morphisms. In particular, to any $\mathcal{O}$-pair (X, F) we can assign its group of automorphisms $Aut(X, F)$.

Let G be a group. By definition an **action** of G on an $\mathcal{O}$ pair (X, F) is a homomorphism $\rho : G \to Aut(X, F)$.

In principle we are interested mostly in cases when G is an algebraic group and the action ρ is algebraic. We will discuss these notions later in more detail.

Definition. A variety X with a distinguished action ρ_X of G we will call a G-space.

2. Fix a a G space X (defined by an action ρ_X). We define a $\mathcal{O}$-module on X to be a $\mathcal{O}$-pair (X, F) equipped with an action ρ of G which defines the action ρ_X on the variety X.

Describe the notion of a morphism between $\mathcal{O}$-modules on a G-space X.

The category of $\mathcal{O}$-modules on a G-space X we denote $\mathcal{M}_G(\mathcal{O}_X)$. Usually objects of this category are called **G-equivariant $\mathcal{O}$-modules on** X.

Remark. A special case of this is an action when the action of the group G on the space X is trivial. In this case we say that G acts on $\mathcal{O}_X$-module F. For example, when $X = pt$ we see that F is a vector space and ρ is just a representation of G.

3. Let $\pi : X \to Y$ be a G-equivariant morphism of algebraic varieties. Define functors $\pi_* : \mathcal{M}_G(\mathcal{O}_X) \to \mathcal{M}_G(\mathcal{O}_Y)$ and $\pi^* : \mathcal{M}_G(\mathcal{O}_Y) \to \mathcal{M}_G(\mathcal{O}_X)$.

II. **Invertible $\mathcal{O}$-modules.**

Definition. An $\mathcal{O}$-module L on an algebraic variety X is called **invertible** if it is locally isomorphic to $\mathcal{O}_X$ as $\mathcal{O}_X$-module.

4. Denote by $Pic(X)$ the set of isomorphism classes of invertible $\mathcal{O}$-modules on X. Show that this set has a natural structure of an abelian group. Show that any morphism $\pi : X \to Y$ induces a homomorphism of groups $\pi^* : Pic(Y) \to Pic(X)$.

III. **Representations of the multiplicative group G_m and gradings.**

We will be mostly interested in the case when $G = k^*$. In fact this is an algebraic group; the standard notation for this group is G_m.

Definition. Fix an algebraic group G (for example $G = G_m$). Let ρ be a representation of the group G_M in a vector space V. It is called **algebraic** in the following cases

(a) If V is finite dimensional we require that all matrix coefficients of ρ are regular functions on G.

(b) In general ρ is called algebraic if V is a union of finite dimensional G-invariant subspaces on each of them the representation is algebraic.

[P] 5. Show that to define an algebraic action of the group G_m on a vector space V is exactly the same as to define a **Z**-grading on V. Namely, to a grading $V = \bigoplus V^k$ corresponds the action ρ of G_m given by $\rho(a)v = a^k v$ for $a \in k^*$, $v \in V^k$.

IV. **G_m-bundles and invertible $\mathcal{O}$-modules.**

Definition. Fix a group G and an algebraic variety S. Consider S as a G-space with the trivial action ρ_S.

A G-pre-bundle on S is a pair (X,p) where X is a G-space and $p : X \to S$ a morphism of G-spaces such that the action of G on X is free and $S = X/G$ as a set. A G-pre-bundle is called a G-bundle if the projection p is locally trivial. The last condition means that X can be covered by open affine subsets U such that the pre-bundle $p : p^{-1}(U) \to U$ is isomorphic to a trivial pre-bundle $pr : G \times X \to U$.

[P] 6. Let $p : X \to S$ be a G_m-bundle on S. Consider an $\mathcal{O}$-module F on S and set $R = p_*(p^*(F))$

Show that R is a G_m-equivariant $\mathcal{O}$-module on S. Show that it has natural grading defined by the action of the group G_m, namely $R = \bigoplus_k R^k$. Deduce from this that the action of the group G_m on the space of global sections $\Gamma(X, p^*(F)*$ is algebraic.

Remark. Here R^k is locally isomorphic to F but might be not isomorphic globally.

[P] 7. Let $p : X \to S$ be a G_m-bundle on S. We can assign to it an invertible $\mathcal{O}_S$-module $\mathcal{O}_S(1)$. Show that this construction gives an equivalence between the category of G_m-bundles on S and the category of invertible $\mathcal{O}_S$-modules (with morphisms being isomorphisms).

V. **Invertible $\mathcal{O}$-modules and projective morphisms.**

Frequently used case of this construction is the following. Let us fix a finite-dimensional vector space V and set $S = \mathbf{P}(V)$. In this case we have a canonical G_m-bundle (X,p) on S, where $X = \mathbf{V}^\times := \mathbf{V} \setminus 0$ and $p : X \to S$ is the canonical projection.

In this case $\mathcal{O}$-modules R^k on S produced from an $\mathcal{O}_S$-module F are called **twists** of F (standard notation for this $\mathcal{O}$-module is $F(k)$).

[P] 8. Show that $F(k) = \mathcal{O}(k) \otimes_{\mathcal{O}_S} F$.

Let $\pi : X \to \mathbf{P}(V)$ be a morphism of algebraic varieties. Then on the variety X we get the following algebraic structure

(Ξ) (i) Invertible $\mathcal{O}$-module L

(ii) Morphism $p : V^* \to \Gamma(X, L)$

This structure satisfies the following condition:

(*) The space V^* generates the $\mathcal{O}$-module L, i.e. for every point $x \in X$ the induced morphism of vector spaces $V^* \to L|_x$ is onto.

Namely we take $L = \pi^*(\mathcal{O}(1)$.

[P] 9. (i) Explain how to construct the structure Ξ from morphism π.

(ii) Show that any algebraic structure Ξ satisfying axiom (*) corresponds to a morphism $\pi : X \to \mathbf{P}(V)$. Show that this gives a bijective correspondence between morphisms and structures Ξ.

This is a deep result since it allows to describe a geometric object - a morphism π - in more or less algebraic terms. It gives a way to produce many non-trivial morphisms of the variety X into projective spaces.