Problem assignment 1.

Representations of reductive p-adic groups.

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A remark on problems in different areas. In my assignments I will try to single out problems that are not directly related to representation theory. For example sign (LA) signifies a problem (or a definition) from linear algebra, (Top) from topology and so on.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Dmitry).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

In this assignment X is an l-space. By definition a **ball** in X is an open compact subset.

1. (Top) (i) Show that every locally closed subset $Y \subset X$ is an l-space.

(ii) Show that every compact subset K of X is contained in a ball.

2. (Top) (i) Consider a compact subset $K \subset X$. Suppose we have a family of open subsets U_κ which covers K (i.e. the union of the subsets U_κ contains K.

Show that there exists a finite family of disjoint balls $B_i \subset X$, $i = 1, ..., r$ which covers K and is inscribed in the family U_κ (i.e. every ball B_i lies in one of the subsets U_κ).

(ii) Suppose that the family of open subsets U_κ covers X. Show that there exists a countable family of disjoint balls B_i which covers X and is inscribed in the family U_κ.

3. Let $U \subset X$ be an open subset and $Z = X \setminus U$ be its closed complement (such pair (U, Z) we will call a small stratification of X).

Construct morphisms $ext : S(U) \to S(X)$ (extension by zero) and $res : S(X) \to S(Z)$ (restriction).

Show that they form the standard short exact sequence $0 \to S(U) \to S(X) \to S(Z) \to 0$.

[**P**] **4.** Define the space of **distributions** $\mathcal{D}(X)$ on the l-space X to be $\mathcal{D}(X) := S(X)^*$. The value of the distribution E on the function f we usually denote $< E, f >$ or $\int fE$.

Construct morphisms of restriction $\mathcal{D}(X) \to \mathcal{D}(U)$ and extension $\mathcal{D}(Z) \to \mathcal{D}(X)$. Show that they form a short exact sequence.

In other words we can (and will) identify the space $\mathcal{D}(Z)$ with the space of distributions on X that vanish on U.

[P] 5. Consider a family of open subsets $U_\kappa \subset X$ and denote by U the union of these subsets.

Consider a distribution $E \in \mathcal{D}(X)$. Show that E vanishes on U (i.e the restriction of E to U is 0) if and only if E vanishes on every set U_κ.

Definition. Consider a distribution $E \in \mathcal{D}(X)$. Show that there exists unique minimal closed subset $S \subset X$ such that the distribution E vanishes on $X \setminus S$. This subset is called the **support** of E (notation $S = Supp(E)$).

[P] 6. Let E be a distribution on X with compact support. Show that the functional E **canonically** extends from the space $S(X)$ to the space $C^\infty(X)$ of all locally constant functions on X.

We will use the same notation $\int fE$ for this extension.

Consider an automorphism A of the l-space X. It induces the automorphisms of the spaces $S(X)$ and $\mathcal{D}(X)$ by formulas

$A(f)(x) := f(A^{-1}(x))$, $< A(E), f >:=< E, A^{-1}(f) >$

These operations are compatible with products. In particular, any action of a group G on an l-space X defines an action of G on spaces $S(X)$ and $\mathcal{D}(X)$.

[P] 7. Let K be a compact l-group. Show that there exists unique distribution E on K which is left invariant (i.e. invariant with respect to the left translation) such that $< E, 1 > = 1$. This distribution is also right invariant. We will denote this distribution by e_K.

[P] 8. (i) Let G be an l-group. Show that there exists unique up to scalar left invariant distribution E on G. Show that it can be chosen to be positive.

This distribution is usually called a **Haar measure** on G; we will denote it μ_G.

(ii) Let A be any group automorphism of the l-group G. Show that then $A(\mu_G) = c\ \mu_G$ for some positive constant c. This constant $c = c(A)$ does not depend on the choice of the Haar measure. We call this constant the **modulus** of the automorphism A and denote it by $\delta(A)$.

In particular, for any element $g \in G$ we can consider the corresponding conjugation automorphism $Ad_g : G \to G$ and define the modulus $\delta(g) := \delta(Ad_g)$.

[P] 9. Let F be a p-adic field. Then the group $H = F^*$ acts on the additive group $G = F$.

Compute the modulus character $\delta : H \to \mathbf{R}^{+*}$ of this action.