Problem assignment 2.

Representations of p-adic groups - Langlands program.

Joseph Bernstein October 29, 2009.

1. Let L/K be a finite extension of number fields (i.e. L and K are finite extensions of $\mathbf{Q}$).

Let w be a place of K and K_w be the corresponding completion of K. Consider a commutative K_w-algebra $L_w = L \otimes_K K_w$.

(i) Show that this algebra is locally compact. Show that it is isomorphic to a direct sum of fields L_v corresponding to all places v of L over w.

(ii) Show that $\sum_{v \mapsto w} \dim L_v = [L : K]$ (here the sum is taken over all places v over w).

2. Let L/K be a finite extension of number fields. Show that for almost all places w of K all places v of L over w are unramified.

3. Let K be a number field, w its place and $F = K_w$ its completion at w. Consider a finite extension E/F. Show that there exists a finite field extension L/K and a valuation v of L over w such that $L_w = E$ and $[L : K] = [F : F]$.

4. Let K be a number field, $\mathbf{A} = \mathbf{A}_K$ its adele ring. Show that

(i) $\mathbf{A}$ is a locally compact ring.

(ii) $K \subset \mathbf{A}$ is a discrete subgroup.

(iii) The space $X(K) := \mathbf{A}_K/K$ is compact.

(iv) Use this to prove the product formula for modulus norms $|\ .$ on K^*.

5. Let $I = I_K := \mathbf{A}_K^*$ be the group of ideles. Consider the group of classes of ideles $Cl(K) := I_K/K^*$.

(i) Construct the modulus morphism $|\ .\ | : I \to \mathbf{R}^+*$ and show that it is equal to the product of local morphisms $|\ .\ |_w$.

(ii) Show that modulus morphism defines a morphism $|\ .\ | : Cl(K) \to \mathbf{R}^+*$.

(iii) Show that the kernel $Cl^1(K)$ of the morphism in (ii) is compact.

6. (i) Show that the subring $\mathcal{O}_K \subset K$ of integral elements is the intersection in K of subrings $\mathcal{O}_w$ for all non-Archimedean valuations w.

(ii) Consider the group $U_K := \mathcal{O}_K^*$ (it is called the group of units of the field K).

Show that this is a finitely generated group and its rank is equal to $r - 1$ where r ia the number of distinct Archimedean places w of K.

7. Let E/F be an unramified extension of local fields, Γ its Galois group.

Define Artin symbol $Art_{E/F} : F^* \to \Gamma$. Show that its kernel equals $N(E^*)$ where $N : E^* \to F^*$ is the norm map.

8. Prove the weak approximation theorem.

9. Let L/K be a finite abelian Galois extension of number fields, $\Gamma := Gal(L/K)$.

(i) Let w be a place in K over which L is unramified. Define the Artin symbol $Art_w : K^* \to \Gamma$ as a product $Art_w = \prod_{v \mapsto w} Art_v$. Show that it is 1 on $N(L^*)$.

(ii) Fix a finite set S of places of K such that the extension L/K is unramified outside S. Construct groups $I_S(K)$ and $I_S(L)$.

Suppose we know that the Artin symbol can be extended to a morphism $Art_S : I_S(K) \to \Gamma$ that satisfies the product formula for all $a \in K^*$

(i) Show that this extension is unique

(ii) Show that $Art_S(N(I_S(L)) = 1$.