Problem assignment 5.

Analysis on Manifolds.

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I. We will be working within the category of Hausdorff locally compact spaces.

Let $\nu : X \to Y$ be a continuous map.

Definition. (i) The map ν is called **closed** if it maps closed subsets into closed ones.

(ii) The map ν is called **proper** if for any compact subset $C \subset Y$ its preimage $\nu^{-1}(C) \subset X$ is compact.

1. (i) Show that a proper map is closed. Moreover show that it is **universally closed** that means that for any space W in our category the product map $\nu_W : X \times W \to Y \times W$ is closed.

(*) (ii) Conversely, show that any universally closed map $\nu : X \to Y$ is proper.

Definition. We say that the map $\nu : X \to Y$ is a **closed topological imbedding** if it gives a homeomorhism of X with a closed subset $F \subset Y$ (we consider induced topology on F).

2. Show that a continuous map $\nu : X \to Y$ is a closed topological imbedding iff it is an imbedding and it is proper.

Remark. The significance of this statement is that these last two conditions are usually easy to verify.

II. Now back to manifolds.

3. Let $\nu : M \to N$ be a morphism of manifolds. Suppose that ν is a topological closed imbedding and that it is immersion.
Show that then ν is a closed manifold imbedding, i.e. ν defines a diffeomorphism of M with a closed submanifold $F \subset N$.

Definition. Let $\nu : M \to N$ be a morphism of manifolds and $Z \subset N$ a submanifold. We say that ν is **transversal to** Z if for any point $a \in M$ either $\nu(a) \notin Z$ or $\nu(a) = b$ lies in Z and $D\nu(T_a(M)) + T_b(Z) = T_b(N)$.

4. (i) Let $Z \subset X \subset Y$ be a system of manifolds. Show that locally it is diffeomorphic to a system of linear spaces.
(ii) Let Z, W be a system of two submanifolds in a manifold X. Show that if they are transversal then locally this system is diffeomorphic to a system of linear spaces.

5. Suppose that a morphism $\phi : X \to Y$ is transversal to a submanifold $Z \subset Y$. Show that then the preimage $W = \phi^{-1}(Z)$ is a submanifold of X of expected dimension. Explain how to compute tangent spaces of W.

Consider also another morphism $\phi' : X \to Y'$ and assume that it is transversal to a submanifold $Z' \subset Y'$, so that $W' = \phi'^{-1}(Z')$ is a submanifold of X.

Show that the restriction of the morphism ϕ to $W' \subset X$ is transversal to Z iff the restriction of the morphism ϕ' to $W \subset X$ is transversal to Z'.

6. Let V be the space of matrices $Mat(m, n)$ of size $m \times n$. For every r consider the subset $M_r \subset V$ of matrices of rank r.

(i) Show that this is a submanifold. Compute its dimension.

(ii) For every point $m \in M_r$ describe the tangent space $T_m(M_r) \subset V$

7. Let M be a manifold of dimension m and $\phi : K \to M$ a morphism of manifolds with $\dim K < m$.

Show directly that the image $\phi(K) \subset M$ has measure 0.

Remark. The example of Peano curve shows that this is not true for **continuous** maps.

8. Construct an example of a smooth morphism $f : \mathbf{R} \to \mathbf{R}$ for which the set of the critical values is dense.

9. Using Sard's lemma show that for a manifold K of dimension $< n$ any morphism $\phi : K \to S^n$ can be contracted to a point.

(We will soon see that this implies that S^n is not diffeomorphic to $S^k \times S^{n-k}$ for $0 < k < n$.)

10. Consider a morphism $\nu : M \times S \to N$ that we also interpret as a **smooth family of morphisms** $\nu_s : M \to N$ parameterized by points s of the base S.

Let $Z \subset N$ be a submanifold. We assume that the morphism ν is transversal to Z and denote by W the submanifold $\nu^{-1}(Z) \subset M \times S$.

Show that for every point $(m, s) \in W$ the following conditions are equivalent

(a) the morphism $\nu_s : M \to N$ is transversal to Z at point m.

(b) the projection morphism $p : W \to S$ is submersive at the point (m, s).

Using Sard's lemma deduce from this that for almost all points $s \in S$ the morphism ν_s is transversal to Z.