Transversality.

I. Let M, N be two manifolds. We would like to study the space $\mathcal{M}or(M,N)$ of morphisms $\nu : M \to N$. One can think about this space as some kind of infinite dimensional manifold. The usual way to study such objects is to consider finite dimensional submanifolds $S \subset \mathcal{M}or(M,N)$.

Terminology. Let $\nu_s : M \to N$ be a family of morphisms parameterized by points s of a manifold S. We say that it is a **smooth family of morphisms** if the corresponding map $\nu : S \times M \to N$ is smooth.

Given such a family ν we can restrict it to any submanifold $T \subset S$; we call this the **restriction of the smooth family of morphisms ν_s to T**.

More generally given any morphism $\kappa : T \to S$ we consider the **base change family of morphisms** $\nu_t := \nu(\kappa(t))$ parameterized by points of T.

Terminology. Consider a morphism $\lambda : M \to N$. A **deformation** of λ is a smooth family of morphisms $\nu_s : M \to N$ parameterized by some manifold S and a point $s_0 \in S$ such that $\nu_{s_0} = \lambda$.

Problem **1.** Consider two deformations ν, ν' of a morphism λ parameterized by manifolds S and T. Then there exists a deformation $\rho : R \times M \to N$ and imbeddings of S and T into R such that ν and ν' are obtained from ρ by restriction and $r_0 = s_0 = t_0$.

II. **Transversality.**

Definition. Let $\nu : M \to N$ be a morphism. We say that ν is **transversal** to a submanifold $Z \subset N$ if for any point $a \in M$ either $\nu(a) \notin Z$ or $b = \nu(a)$ lies in Z and the natural morphism $D\nu : T_a(M) \to T_b(N)/T_b(Z)$ is epimorphic.

Problem **2.** Show that if ν is transversal to Z then the subset $W = \nu^{-1}(Z)$ is a submanifold in M of expected dimension. Compute the tangent spaces of points $a \in W$.

In fact it is more convenient to work with slightly more general notion.

Terminology. A **bordism** in a manifold N is a pair (Z, κ), where Z is a manifold and $\kappa : Z \to N$ a morphism of manifolds.

Definition. A morphism $\nu : M \to N$ is **transversal** to a bordism $\kappa : Z \to N$ if for any point $(m, z) \in M \times Z$ either $\nu(m) \neq \kappa(z)$ or $\nu(m) = \kappa(z) = b$ and the subspaces $D\nu(T_m(M))$ and $D\kappa(T_z(Z))$ span the space $T_b(N)$.

Problem **3.** Suppose that a morphism $\nu : M \to N$ is transversal to a bordism $\kappa : Z \to N$. Show that the subset $W = \{(m, z) \in M \times Z |\ \nu(m) = \kappa(z)\}$ is a submanifold of expected dimension. Explain how to compute its tangent spaces.

Terminology. We will call the manifold W of this form a **preimage** of the bordism κ under morphism ν and denote it $\nu^{-1}(\kappa)$.

III. **Sard's lemma.**

Terminology. Let S be a manifold and $X \subset S$ arbitrary subset. We say that X is **m-small** if its measure is 0. We say that X is **m-large** if its complement is m-small.

Problem **4.** (i) Show that union of a countable family of m-small subsets is m-small.

(ii) Show that an m-large subset is always dense.

Definition. Let $\nu : M \to N$ be a morphism of manifolds. A point $a \in M$ is called **critical** (for morphism ν) if the morphism of tangent spaces $D\nu : T_a(M) \to T_{\nu(a)}(N)$ is not onto. It is called **regular** if this morphism of linear spaces is onto.

Terminology. (i) We denote by $Crit(\nu) \subset M$ the closed subset of critical points and by $Reg(\nu) \subset M$ the open subset of regular points.

(ii) The subset $\nu(Cri(\nu) \subset N$ is called **the set of critical values of** ν - natation $Crit - val(\nu)$. Its complement in N is called the set of **regular values of** ν - notation $Reg - val(\nu)$.

Sard's Theorem. For any morphisms $\nu : M \to N$ the subset of critical values $Crit - val(\nu)$ is m-small in N. The subset $Reg - val(\nu)$ of regular values is m-large in N. In particular this subset is dense in N.

IV. **Generically transversal family of morphisms.** Let $\nu_s : M \to N$ be a smooth family of morphisms parameterized by points of a manifold S. We say that this family is **generically transversal** to a bordism $\kappa : Z \to N$ if the corresponding morphism $\nu : S \times M \to N$ is transversal to κ.

Problem **5.** Let $\nu_s : M \to N$ be a smooth family of morphisms parameterized by points of a manifold S that is generically transversal to a bordism $\kappa : Z \to N$. Show that the set X of points $s \in S$ such that morphism ν_s is transversal to κ is m-large and hence dense.

Hint. Show that X is the set of regular values of the projection $p : W \to M$, where $W = \nu^{-1}(Z)$.

V. Existence of strongly submersive families of morphisms.

Definition. A family of morphisms $\nu : S \times M \to N$ is called **strongly submersive** if for any point $m \in M$ the morphism $\nu_m : S = S \times m \to N$ is submersive.

This is a strong condition. It immediately implies that for any submanifold $M' \subset M$ and any bordism $\kappa : Z \to N$ the restriction of the family ν to submanifold M' is generically transversal to Z and hence for many points $s \in S$ the morphism $\nu_s|_{M'}$ is transversal to κ.

Problem **6.** Show that any morphism $\lambda : M \to N$ can be deformed in a strongly submersive family. Deduce that any family of morphisms might be included into a strongly submersive family.

Hint. First show this for the case when $N = U$ is an open subset of an Euclidean space $\mathbf{R}^n$. In this case take S to be unit ball in $\mathbf{R}^n$ and define $\nu(s, m) = \lambda(m) + \phi(m)s$, where ϕ is a smooth strictly positive function on M such that $\phi(m)S \subset U$ for all $m \in M$.

In general case imbed N into $\mathbf{R}^n$ and find a neighborhood U of N that submersively retracts on N.

VI. **Orientation.**

We can repeat all the previous constructions for oriented manifolds (and bordisms). Check that all manifolds and bordisms we construct in this way are naturally oriented.

VII. Intersection theory.

For simplicity consider the case when all manifolds are compact. Fix a bordism $\kappa : Z \to N$. For any morphism $\nu : M \to N$ we would like to understand to what extent we can define the induced morphism $\nu^{-1}(\kappa)$,

The strategy is as follows:

(i) Choose a small deformation ν_0 of ν that is transversal to κ and consider the bordism $\rho_0 = \nu_0^{-1}(\kappa)$.

(ii) If ν_1 is another small deformation of ν transversal to κ we consider the bordism $\rho_1 = \nu_1^{-1}(\kappa$.

The question is how much can differ ρ_0 and ρ_1. The claim is that they differ by a boundary.

Definition. A bordism $\rho : W \to M$ is called a **boundary** (another terminology - ρ is bordant to 0) if there exists an oriented manifold X with boundary and a morphism $\tau : X \to M$ such that the pair $\partial X, \tau$ is isomorphic to the pair W, ρ

(iii) **Claim.** In situation described in (i) and (2) bordisms ρ_1 and ρ_0 are bordant, i.e. their difference is bordant to 0.

The idea is that ν_1 and ν_0 are close and hence homotopic. Thus we can find a smooth family γ of morphisms γ_t parameterized by points of the unit interval I that connects ν_0 with ν_1.

Moreover, we can deform the family γ in such a way that it is transversal to κ (and still connects ν_0 and ν_1.

Now we can take X to be $\gamma^{-1}(\kappa)$. The general transversality results imply that $\partial X = \rho_1 - \rho_0$

Remarks. 1. We have developed this idea in class in case when M and Z have complementary dimensions in N. In this case ρ_i are bordism of dimension 0 and they are essentially defined by their degree.

Think how to work this out in more general case.

2. Let us denote by $Bor^l(N)$ the abelian group generated by all bordisms in N of **codimension** l and relations that any morphism bordant to 0 gives 0 in $Bor^l(N)$.

Show that the construction described above defines a natural morphism $\nu^* : Bor^l(N) \to Bor^l(M)$.

VII. In order to carry out the procedure described above we need a result that strengthens the result of Problem 6.

Problem **7.** Consider a morphism $\lambda : M \to N$ and a bordism $\kappa : Z \to N$. Suppose we found a closed subset $C \subset M$ such that λ is transversal to κ at all points of C.

Show that there exists a deformation ν_s of morphism λ such that

(i) $\nu_s|_C \equiv \lambda|_C$

(ii) The family ν_s is strongly submersive on the open subset $M_0 = M \setminus C$

Hint. Find a smooth function α on M with values in $I = [0, 1]$ such that its set of zeroes is exactly C. Then modify the construction in Problem 6 as follows $\nu(s, m) = \lambda(m) + \alpha(m)\phi(m)s$.