Differential forms and DeRham complex.

I Differential forms.

Definition. Given a vector space V we denote by $Alt^k(V)$ the space of skew-symmetric multilinear forms on V in k variables.

The space $Alt(V) = \oplus Alt^k(V)$ has natural structure of an $\mathbf{R}$-algebra.

Definition. Let U be an abstract domain. For a fixed k consider a family of vector spaces $Alt^k(T_xU)$ parameterized by points $x \in U$. A section ω of this family (i.e a correspondence $x \mapsto \omega_x \in Alt^k(T_xU)$) we call a general differential k-form. Such form is called **smooth** if for any collection of smooth vector fields $\xi_1, ..., \xi_k$ the function $\omega(\xi_1, ..., \xi_k)$ is smooth.

All differential forms we will consider are assumed to be smooth. The space of k-forms we denote by $\Omega^k(U)$. The space $\Omega(U) = \oplus_k \Omega^k(U)$ is a graded algebra. This algebra is a commutative super algebra, i.e. it satisfies $\omega\eta = (-1)^{deg\omega \deg\eta}\eta\omega$.

Problem **1.** Show the following facts:

(i) $\Omega^0(U) = S(U)$

(ii) $\Omega^1(U)$ a free $S(U)$-module of rank $n = \dim U$. It is dual to the module $Vect(U)$. Write a basis of this module choosing coordinates on U.

(iii) Write a basis of an $S(U)$-module $\Omega(U)$ choosing coordinates on U.

II DeRham differential. Let U be an abstract domain. We define a linear map $d : S(U) = \Omega^0(U) \to \Omega^1(U)$ by $df(\xi) = \xi(f),\ f \in S(U), \xi \in Vect(U)$.

Theorem. There exists unique operator $d : \Omega(U) \to \Omega(U)$ that satisfies the following conditions:

(i) d is a derivation of (super) algebra $\Omega(U)$ of degree 1, i.e. it satisfies the Leibnitz rule $d(\omega\eta) = (d\omega)\eta + (-1)^{deg\omega}\omega(d\eta)$.

(ii) $d^2 = 0$

(iii) On $\Omega^0(U)$ d is the operator described above.

Remark. In general I always prefer a definition of an object that is not a construction but that defines it by some properties. It is usually formulated as a statement the there exists unique object satisfying some conditions.

As a rule of good manners in such cases the proof should start with the proof of uniqueness, and only after this existence.

Proof. (i) Uniqueness. Choose coordinates (x_i) and consider their differentials $dx_i \in \Omega^1(U)$. Then $d(dx_i) = 0$.

Since monomials dx^α form a basis of $\Omega(U)$ as $S(U)$-module using Leibnitz rule we get explicit formula for d,

(*) $d(\sum f_\alpha dx^\alpha) = \sum d(f_\alpha) dx^\alpha$

(ii) Existence. Formula (*) defines an operator d satisfying all conditions.

Weyl's formulas.

For any vector field ξ we define two operators: inner multiplication i_ξ : $\Omega(M) \to \Omega(M)$ and Lie derivative $L_\xi : \Omega(M) \to \Omega(M)$.

Operator i_ξ is a derivation of the algebra $\Omega(M)$ of degree -1 (i,e, it satisfies the Leibnitz rule $i_\xi(\omega\eta) = i_x i(\omega)\eta + (-1)^{deg(\omega)}\omega d\eta)$ and on 1-forms it is given by $\imath_\xi(\alpha) = \alpha(\xi)$.

Operator L_ξ is defined by $L_\xi = [d, i_\xi] = di_\xi + i_\xi d$ (pay attention to signs).

Problem **2.** Show Weyl formulas for these operators

(i) d, i_ξ, L_ξ are derivations of the algebra $\Omega(M)$.
(ii) $d^2 = 0$; $i_\xi^2 = 0$ and hence $[i_\xi, i_\eta] = 0$.
(iii) $[d, L_\xi] = 0$; on $\Omega^0(M)$ operator L_ξ coincides with ξ.
(iv) $[L_\xi, i_\eta] = i_{[\xi,\eta]}$, $[L_\xi, L_\eta] = L_{[\xi,\eta]}$

Important corollary. Let $n = \dim U$. Then for top forms $\omega \in \Omega^n(U)$ we have an identity $L_{f\xi}(\omega) = L_\xi(f\omega)$.