Integration of differential forms.

Let U be an abstract domain. We denote by $V(U)$ the space $\Omega_c^{top}(U)$ of differential forms of top degree with compact support in U. We denote by $V'(U) \subset V(U)$ be the subspace of Lie derivatives (i.e. $V'(U) = \sum_{\xi \in Vect(U)} L_\xi(V(U))$) and by $Q(U)$ the quotient space $V(U)/V'(U)$.

Theorem. Suppose the domain U is connected. Then the space $Q(U)$ is 1-dimensional.

Proof. Let us introduce coordinates (x_i) and consider partial derivatives ∂_i. The corollary to Weyl's formulas shows that $V'(U) = \sum_i L_{\partial_i} V(U)$.

Step 1. **Lemma.** (i) Suppose C is an open coordinate cube. Then the space $Q(C)$ is one dimensional and non-trivial functional on it is given by an integral $\int_{(x)}$ (here integral might be a Riemann integral or just a repeated integral).

(ii) If D is another coordinate cube that lies in C then the natural morphism $Q(D) \to Q(C)$ is an isomorphism.

The proof of (i) for dimension 1 is a straightforward application of integration in dimension 1. Using smooth dependence of 1-dimensional integral on parameters we can reduce the proof of (i) in dimension n to dimension $n-1$ and hence prove it by induction.

Item (ii) immediately follows from (i).

Step 2. The lemma in step 1 implies that the space $Q(\mathbf{R}^n)$ is 1-dimensional. This implies that for any open coordinate cube $C \subset U$ the image of the space $V(C)$ in $Q(U)$ is 1-dimensional (analyze morphisms $Q(C) \to Q(U) \to Q(\mathbf{R}^n)$). We denote this subspace (this line) by $l(C)$.

Step 3. If two cubes $C, D \subset U$ intersect then $l(C) = l(D)$.

Enough to consider the case $D \subset C$ which follows from Step 1.

Step 4. For any line l in $Q(U)$ consider the union U_l of all cubes $C \subset U$ such that $l(C) = l$. This gives a decomplsition of U into open disjoint subsets. Since U is connected it coincides with U_l for some specific l.

Step 5. We have shown that U can be covered by open subcubes C such that $l(C) = l$. Partition of unity then implies that $Q(U) = l$, QED.

Corollary. If we have two coordinate systems (x) and (y) on U then there exists a constant c such that on the space $V(U)$ we have an identity

$\int_{(x)} \rho = c \int_{(y)} \rho$

Now it is not difficult to check that the constant c is this formula is 1 or -1 depending on orientations of coordinate systems $(x), (y)$.

Remark. In particular this proves the formula for change of variables in an integral.