Tangent spaces and vector fields.

Let (X, S(X)) be an abstract domain.

I claim that to every point $a \in X$ canonically corresponds some vector space T_aX (this space is called the tangent space to the domain X at the point a). We will see that for different points of X these spaces are different and in some sense are not related (though there is some connection between them).

I will give two descriptions of this space T_aX – one geometric and one algebraic. This is typical for differential geometry – we will usually have equivalent descriptions of the objects we study - geometric and algebraic. Sometimes it is easier to work with one, sometimes with another. Many notions are easier to formulate in algebraic picture. They usually can be also interpreted geometrically, but this interpretation might be quite sophisticated.

Geometric picture. We are given an abstract domain (X, S(X)) and a point $a \in X$. Define a **short curve** as a smooth morphism $\gamma : \mathbf{R} \to X$ defined in some neighborhood of the point 0 such that $\gamma(0) = a$.

Consider the set SC of all short curves. We introduce an equivalence relation on this set. Namely we say that $\gamma_1 \sim \gamma_2$ if for every function $f \in S(X)$ the functions $\gamma_1^*(f)$ and $\gamma_2^*(f)$ in $S(\mathbf{R})$ are close at 0, i.e. $\gamma_1^*(f) - \gamma_2^*(f) = o(t)$.

Problem **1.** Consider a coordinate system on X. It allows to introduce a distance d between points of X. Show that two curves γ_1, γ_2 are equivalent iff $d(\gamma_1(t), \gamma_2(t)) = o(t)$.

Definition. The tangent vector ξ to X at the point a is a short curve up to above equivalence relation.

The set of tangent vectors we denote by $T_a^{geom}X$.

Algebraic picture.

Definition. Let a be a point of X. An a-derivation of the algebra $S(X)$ is a linear morphism $D : S(X) \to \mathbf{R}$ that satisfies the Leibnitz rule

$D(fh) = D(f)h(a) + f(a)D(h)$

All a-derivations form a linear space $Der_a(S(X))$. This is an algebraic version of the tangent space; we will denote it $T_a^{alg}X$.

I claim that the set $T_a^{geom}X$ is canonically in bijection with the set $T_a^{alg}X = Der_a(S(X))$. Let us construct the corresponding map of sets.

Given a short curve γ we define the derivation $D_\gamma : S(X) \to \mathbf{R}$ by

$D_\gamma(f) := \frac{d}{dt}\gamma^*(f)|_{t=0}$

Problem **2.** Check that

(i) $D_\gamma \in Der_a(S(X))$

(ii) $D_{\gamma_1} = D_{\gamma_2}$ iff $\gamma_1 \sim \gamma_2$

Proposition. The map $T_a^{geom}X \to Der_a(S(X))$ is a bijection.

Note that we constructed both sets and the map between them without choosing any coordinate system. Now in order to prove the proposition we will introduce a coordinate system (x^i) on X (we assume the coordinates vanish at the point a).

Consider the operators $\partial_i : S(X) \to \mathbf{R}$ defined by $\partial_i(f) = \frac{\partial f}{\partial x^i}(a)$.

Claim. Morphisms ∂_i lie in $D_a(S(X))$ and form a basis of this space.

This claim easily follows from

Hadamard's Lemma. Any function $f \in S(X)$ can be written as $f = f(0) + \sum b_i x^i + \sum x^i h_i$ where b_i are constants and h_i are functions such that $h_i(0) = 0$.

Problem **3.** Prove the claim above using Hadamard's lemma and explain how it implies the proposition the proposition.

Vector fields.

Definition. A vector field ξ is a collection of tangent vectors ξ_a for all points $a \in X$.

Such vector field ξ defines a linear morphism $\xi : S(X) \to F(X)$ (the space of all functions on X). We say that ξ is **smooth** is $\xi(S(X)) \subset S(X)$.

In coordinates we can write $\xi_x = \sum a^i(x)\partial_i$, i.e. $\xi(f) = \sum a^i \frac{\partial f}{\partial x^i}$. The vector field is smooth iff the coordinate functions a^i are smooth.

Since $a^i = \xi(x^i)$ we see that a vector field ξ is smooth iff $\xi(x^i) \in S(X)$ for coordinate functions x^i.

We denote by $Vect(X)$ the space of smooth vector fields on X. This is one of the central objects of differential geometry.

Let us give a purely algebraic description of this space.

Definition. (from algebra). Let k be a field and A any k-algebra. This means that A is a k-vector space equipped with a bilinear multiplication $m : A \times A \to A$. So far we do not impose other restrictions on m.

A k-**derivation** of the algebra A is a k-linear morphism $D : A \to A$ that satisfies the Leibnitz rule

$D(m(a, b)) = m(Da, b) + m(a, Db)$.

The k linear space of such derivations we denote $Der_k(A)$.

It is clear from above that $Vect(X) = Der_k(S(X))$.

Problem **4.** Let $U \subset X$ be an open subdomain.

(i) Show that for any point $a \in U$ the tangent spaces T_aU and T_aX are canonically isomorphic.

(ii) Define the restriction morphism $res_{U,X} : Vect(X) \to Vect(U)$. Namely show that for any vector field ξ on X there exists unique vector field η on U such that for any function $f \in S(X)$ we have $\xi(f)|_U = \eta(f|_U)$.

This shows that notions of tangent space and of a vector field are local in nature.

Lie algebra structure on vector fields.

From the algebraic description we can discover a new structure on the space of vector fields – the structure of a Lie algebra. This is a very important structure. We will also explain the geometric meaning of this structure, but this will be much later.

Lemma (from algebra). Let A be a k-algebra, $D_1, D_2 \in Der_k(A)$ its derivation. Consider the operator $D : A \to A$ defined by $D = D_1D_2 - D_2D_1$ (standard notation for this operation is $D = [D_1, D_2]$; the operator D is called the **commutator** or **bracket** of operators D_1 and D_2).

Then D is a k-derivation of the algebra A.

The proof is a straightforward calculation.

Thus we see that the space $Vect(X)$ of vector fields on an abstract domain X has an additional operation – (**bracket** or **commutator** of vector fields). It is a Lie algebra with respect to bracket, i.e. this operation satisfies Jacobi identity.

Problem **5.** a) Show the following properties of the commutator

(i) $[\xi, \eta]$ is skew symmetric in ξ, η and satisfies the Jacobi identity
$[\xi, [\eta, \theta]] + [\theta, [\xi, \eta]] + [\eta, [\theta, \xi]] = 0$

(ii) $\xi(fh) = f\xi(h) + \xi(f)h$, $(f\xi)(h) = f(\xi(h))$, $[\xi, f\eta] = \xi(f)\eta + f[\xi, \eta]$

(b) Write down explicitly bracket operation for vector fields in coordinates.