Analysis on Manifolds.

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Problems about metric spaces.

Definition. Let $\nu : X \to Y$ be a map of two metric spaces, $x \in X$ and $y = \nu(a) \in Y$. The map ν is called **continuous** at the point x if for any sequence of points $x_i \in X$ convergent to x their images $\nu(x_i) \in Y$ converge to y.

The map ν is called continuous if it is continuous at all points $x \in X$.

1. Let $\nu : X \to Y$ be a map of two metric spaces.

(i) Consider a point $x \in X$ and its image $y = \nu(x) \in B$. Show that ν is continuous at the point x iff it satisfies:

(*) For any neighborhood V 0f y in Y the subset $\nu^{-1}(V) \subset X$ is a neighborhood of x in X.

(ii) Show that ν is continuous iff it satisfies

(**) For any open subset $V \subset Y$ the subset $\nu^{-1}(V) \subset X$ is open in X.

2. Let X be a metric space. Show that a map $f : X \to \mathbf{R}^n$ is continuous iff its coordinate functions $f^i \quad (i = 1, ..., n)$ are continuous.

Definition. Let X be metric space. We say that X is **compact** if any sequence of points $x_i \in X$ has a subsequence that converges to some point $a \in X$.

3. Let X be a compact metric space.

(i) Show that any closed subset $Z \subset X$ is compact (with respect to induced metric).

(ii) Show that for any continuous map $\nu : X \to Y$ from X to a metric space Y the image $\nu(X) \subset Y$ is a closed subset compact in induced metric.

4. Let f be a continuous function on a compact metric space X. Show that it is bounded and takes its maximal value.

5. Let $X \subset \mathbf{R}^n$ be a compact subset. Show that X is bounded and closed subset of $\mathbf{R}^n$.

Conversely, show that any closed bounded subset $X \subset \mathbf{R}^n$ is compact.

6. Let X be a metric space. Show that it is compact iff it satisfies the following

Finite Covering Property. Any open covering $\{U_\alpha\}$ contains a finite subcovering $\{U_{\alpha_i}\}$.

7. Let X be a compact metric space and $\{F_\alpha\}$ a family of closed subsets of X . Suppose we know that any finite collection of these subsets has non-empty intersection. Show that all these subsets have non-empty intersection.

8. Let $A, B \subset X$ be two non-empty subsets of a metric space X. We define the distance $d(A, B)$ by $d(A, B) = \inf(d(a, b)|a \in A, b \in B)$.

(i) Show that if A is a set consisting of one point a then $d(A, B) = 0$ iff a lies in the closure of the set B.

(ii) Suppose $X = \mathbf{R}^n$, A is closed and B is compact. Show that there exist points $a \in A$ and $b \in B$ such that $d(a, b) = d(A, B)$.

(iii) Construct example of two closed subsets $A, B \subset \mathbf{R}^n$ such that they do not intersect but $d(A, B) = 0$.

9. Let C be a compact subset of a metric space X and $U \subset X$ be an open subset which contains C. Show that there exists $\varepsilon > 0$ such that U contains the ε -neighborhood of C.

10. Let $\nu : X \to Y$ be a continuous map of two metric spaces. Suppose X is compact. Show that ν is uniformly continuous, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any two points $x, x' \in X$ with $d(x, x') < \delta$ we have $d(\nu(x), \nu(x')) < \varepsilon$.