Problems in linear algebra.

Analysis on Manifolds.

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1. Let $A : V \to W$ be a morphism of vector spaces, $K = \ker A$ its kernel and $I = ImA$ its image.

Show that $K \subset V$ and $I \subset W$ are linear subspaces.

Show that A is mono iff $K = 0$. Show that if $K = 0$ and $I = W$ then A is an isomorphism, i.e. there exists an inverse morphism $B : W \to V$ such that compositions $A \circ B$ and $B \circ A$ are identity morphisms.

2. Let V be a vector space and $L \subset V$ a subspace. Show that there exists a vector space Q and an epimorphism $p : V \to Q$ such that $\ker p = L$.

Show that the pair (Q, p) is uniquely defined up to canonical isomorphism (i.e. any two such pairs are canonically isomorphic).

The space Q is called the **quotient space**; usually it is denoted by V/L.

3. Let V be vector space of dimension $n < \infty$ and $L \subset V$ be a subspace of dimension l. Show that there exists a basis $e_1, ..., e_n$ of the space V such that vectors $e_1, ..., e_l$ form a basis of L.

Show that in this case the vectors $e_{l+1}, ..., e_n$ (or more precisely their images) form a basis of the quotient space V/L.

4. Let V be a vector space of dimension n, $L, L' \subset V$ subspaces of V. Show that if $\dim L + \dim L' > n$ then L and L' have a non-zero intersection.

5. Let V be a vector space of dimension n and $L \subset V$ a subspace. Consider its orthogonal complement $L^\perp \subset V^*$ defined by $L^\perp := \{f \in V^* | f|L = 0\}$.

(i) What is the dimension of $L^\perp$?

(ii) Show that $(R \cap L)^\perp = R^\perp + L^\perp$ and $(R + L)^\perp = R^\perp \cap L^\perp$.

(iii) Show that $(L^\perp)^\perp = L$.

(iv) Show that $L^\perp$ is naturally isomorphic to $(V/L)^*$.

6. Let V be a vector space over $\mathbf{R}$, (e_i) a basis of V and $(x^i) = (x^1, ...x^n)$ the corresponding system of coordinates on V.

(i) How to describe a vector $v \in V$ in this coordinate system ?

How to describe a covector $\xi \in V^*$?

How to describe an endomorphism $A : V \to V$?

How to describe a biliner form B ?

(ii) Let (f_j) be another basis, $C = (c^i_j) \in Mat(n, \mathbf{R})$ the transformation matrix from basis e to basis f (i.e $f_j = \sum c^i_j e_i$). We denote by $D = (d^j_i)$ the inverse matrix.

Describe how coordinates of vectors, covectors, endomorphisms and bilinear forms change from basis e to basis f.

7. Let B be a symmetric bilinear form on V. Denote by Q the corresponding quadratic form on V defined by $Q(x) = B(x, x)$.

(i) Show that the form B could be recovered from Q.

(ii) Show that a function Q on V is a quadratic form iff in any coordinate system it could be written as $\sum a_{ij}x^i x^j$.

(iii) Show that Q is a quadratic form iff it is homogeneous function of degree 2 which for any $a, b \in V$ satisfies the condition that the function $Q(x+a+b) - Q(x+a) - Q(x+b) + Q(x)$ is constant function.

8. Let Q be a quadratic form on a vector space V of dimension n.

(i) Show that one can choose coordinate system (x^i) on V in which the form Q is diagonal, i.e. $Q(x) = \sum a_i(x^i)^2$

(ii) Show that the number of zeroes in diagonal coefficients (a_i) is an invariant of the form Q (i.e. it does not depend on a choice of the coordinate system).

Namely, show that it is equal to $\dim K$ where K is the **kernel** of the quadratic form Q, i.e. K is the kernel of the corresponding bilinear form $B : V \to V^*$.

(iii) We will call a subspace $L \subset V$ strictly Q-positive if the restriction of the form Q to this subspace is **positive definite**. Similarly we define the notion of strictly Q-negative subspace.

Show that the number of positive coefficients a_i equals to the maximum dimension of strictly Q-positive subspaces $L \subset V$ (and similarly for negative coefficients).

In particular these numbers are invariants of the form Q.

9. Let (V, Q) be a finite dimensional Euclidean space $E = (V, Q_0)$.

(i) Show that it is isomorphic to $(\mathbf{R}^n, Q_0)$, where Q_0 is the standard quadratic form $Q_0(x_1, ..., x_n) = \sum(x^i)^2$.

(ii) Let Q be a quadratic form on an Euclidean space $E = (V, Q_0)$. Show that there exists a constant $C > 0$ such that $|Q(x)| \leq CQ_0(x)$ for all $x \in V$.

10. Prove the following **Spectral Theorem.**

Let Q be a quadratic form on an Euclidean space $E = (V, Q_0)$. Then there exists a coordinate system (x^i) in which both quadratic forms are diagonal

$Q_0(x) = \sum(x^i)^2$ and $Q(x) = \sum a_i(x^i)^2$.