Vector bundles and connections.

I. **Vector bundles.**

Let M be a manifold.

1. Give the definition of a vector bundle E on M. Define sub-bundle $L \subset E$ and the quotient bundle $N = E/L$.

We will denote by $S(M, E)$ the $S(M)$-module of smooth sections of the bundle E.

Problem 1. (i) Show that any vector bundle E on M has a positive definite metric.

(ii) Show that for any sub-bundle $L \subset E$ one can find a splitting, i.e. an isomorphism $E \approx L \bigoplus N$ where $N = E/L$.

2. Given vector bundles E, F on M describe construction of new vector bundles E^*, $E \otimes F$, $\mathcal{H}om(E, F)$, $\mathcal{E}nd(E)$, $Alt(E)$ and so on; these operations we call *tensor constructions.*

3. Describe the $\Omega(M)$-module $\Omega(M, E)$ of differential forms on M with values in a vector bundle E.

Given a bilinear pairing $B : E \times F \to D$ of vector bundles on M show how to extend it to a pairing of $\Omega(M)$-modules $\Omega(M, E) \times \Omega(M, F) \to \Omega(M, D)$

4. Let $\nu : M \to N$ be a morphism of manifolds.

For any bundle F on M describe the induced bundle $E = \nu^*(F)$ on M.

Show that this operation is functorial and compatible with tensor constructions and pairings.

Describe canonical morphisms $S(N, F) \to S(M,\ \nu^*(F))$ and $\Omega(N, F) \to \Omega(M,\ \nu^*(F))$.

Show that $(\nu\mu)^*$ is canonically isomorphic to $\mu^*\nu^*$.

II. **Connections on a vector bundle.**

5. Let E be a vector bundle on M. A **connection** on E is a linear morphism $\nabla : S(M, E) \to \Omega^1(M, E)$ that satisfies the Leibnitz rule

$\nabla(fu) = df\ u + f\nabla(u)$

Remark. Equivalent way of describing a connection on E is by giving a collection of operators $\nabla_\xi : S(M, E) \to S(M, E)$ corresponding to vector fields ξ on M (operators of covariant differentiation). List the axiomatic properties of these operators that make this collection of operators a connection.

Problem 5.1. (i) Show that the space of all connections on E is an affine space over the linear space $\Omega^1(M, \mathcal{E}nd(E))$.

(ii) Show that on a trivial vector bundle $E = \mathbf{R}^n$ any connection can be written in a standard form $\nabla = d + \alpha$, where $\alpha \in \Omega^1(Mat(n, \mathbf{R}))$.

(iii) Show that for every vector bundle E the set of connections on E is not empty.

Problem 5.2. Show how given connections on vector bundles E, F one can define in canonical way connections on the vector bundles constructed from E and F using tensor constructions.

Problem 5.3 Let $\nu : M \to N$ be a morphism of manifolds, F a vector bundle on N and $E = \nu^*(F)$. Show that for any connection ∇_F on F there exists unique connection ∇_E on E that satisfies $\nabla_E(\nu^*(u)) = \nu^*(\nabla_F(u))$ for all sections $u \in S(N, F)$.

This connection ∇_E is called induced connection; we will denote it $\nu^*(\nabla_F)$.

Problem 5.4. Let ∇ be a connection on a vector bundle E on M. Let L be any sub-bundle of E and $N = E/L$ the quotient bundle.

Show that the connection ∇ defines a morphism of $S(M)$ -modules $\alpha_\nabla : S(M, L) \to \Omega^1(M, N)$ and hence a morphism of vector bundles $\alpha : L \to \mathcal{T}^*(M) \otimes N$.

Show that the connection ∇ induces a connection on the sub-bundle L iff this morphism α vanishes.

Problem 5.5. Suppose that the vector bundle E is decomposed as a direct sum $E = L \oplus N$. Using projection show that any connection ∇ on E defines a connection $pr_L(\nabla)$ on L.

Problem 5.6. Let E be a vector bundle with an Euclidean structure given by a positive definite form B. Let ∇ be a connection on E preserving the form B.

For any sub-bundle $L \subset E$ define the projection connection $\nabla_L = pr_L(\nabla)$ on L.

Prove that this connection ∇_L preserves the restriction B_L of the form B to the sub-bundle L.

III. **Extension of a connection to the higher degree forms.**

6. Let ∇ be a connection on a vector bundle E on M. Show that ∇ uniquely extends to the linear morphism $\nabla : \Omega(M, E) \to \Omega(M, E)$ of degree 1 that satisfies the Leibnitz rule

$$\nabla(\alpha\omega) = d\alpha\ \omega + (-1)^{deg(\alpha)}\alpha\ \nabla(\omega) \quad \text{for } \alpha \in \Omega(M),\ \omega \in \Omega(M, E).$$

From now on by connection on a vector bundle E we mean this morphism $\nabla : \Omega(M, E) \to \Omega(M, E)$.

Problem 6. Show that this extension of connection is compatible with taking induced connections (see Problem 5.3).

IV. **Curvature of a connection.**

7. Given a connection $\nabla : \Omega(M, E) \to \Omega(M, E)$ on a vector bundle E consider the operator of degree 2 $\quad R = \nabla^2 : \Omega(M, E) \to \Omega(M, E)$. Show that this operator is a morphism of $\Omega(M)$-modules. In particular show that it is defined by multiplication by some form $R = R(\nabla) \in \Omega^2(\mathcal{E}nd(E))$.

This form $R(\nabla)$ is called the **curvature** of the connection ∇.

Problem 7.1. Show that in case of trivial bundle $\mathbf{R}^n$ and connection $\nabla = d + \alpha$ we have $R(\nabla) = d\alpha + \alpha \cdot \alpha$.

Problem 7.2. Show that the curvature form $R = R(\nabla)$ can be computed as follows: $R(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}$ for vector fields ξ, η.

Problem 7.3. Explain how to compute connection in bundles obtained by tensor constructions.

Problem 7.4. Show that the operation of taking the curvature of the connection is compatible with induction of vector bundles and connections.

Problem 7.5. Prove the Bianchi identity $\nabla(R(\nabla)) = 0$.

V. **Characteristic classes.**

8. Let E be a vector bundle on M, ∇ a connection on E and $R = R(\nabla) \in \Omega^2(\mathcal{E}nd(E))$ the corresponding curvature form.

Given a homogeneous polynomial Q of degree k on $Mat(n, \mathbf{R})$ that is conjugation invariant describe the form $c_Q = c_Q(R) \in \Omega^{2k}(M)$ that evaluates the polynomial Q on the form R.

Problem 8.1. (i) Show that the form $c_Q(R)$ is closed.

(ii) Show that if ∇' is another connection and $R' = R(\nabla')$ then the forms R, R' are homologous.

Thus we see that the form $c_Q(R)$ defines a DeRham cohomology class in $H^{2k}(M)$ and this class depends just on the vector bundle E and does not depend on the specific choice of the connection.

This class is called the **characteristic class** of the bundle E corresponding to a polynomial Q.

VI. **Affine connections.**

9. Consider the case when $E = \mathcal{T}(M)$ - the tangent bundle of M. A connection ∇ on this bundle is called an **affine connection**. It defines connections on all bundles obtained from $\mathcal{T}(M)$ by tensor constructions.

Let us denote by σ the canonical 1-form $\sigma \in \Omega^1(M, E)$ that is defined in this case.

Given an affine connection ∇ we define its **torsion** form $T(\nabla) \in \Omega^2(E)$ by $T(\nabla) := \nabla(\sigma)$.

We say that an affine connection is torsion free if its torsion form is 0.

Problem 9. Let ∇ be a torsion free affine connection and R its curvature. Show that the 3-form $R \cdot \sigma \in \Omega^3(E)$ vanish.

VII. Levi-Civita connection.

10. Let M be a manifold and B a Riemannian metric on M. Show that there exists unique affine connection ∇ on M that satisfies

(i) ∇ preserves the form B.

(ii) ∇ is torsion free.

This canonical connection on the Riemannian manifold M is called **Levi Civita connection**. This connection ∇ and its curvature form $R = R(\nabla)$ play the central role in Riemannian geometry.

Problem 10. Let M be a Riemannian manifold with Riemannian metric B. Let $N \subset M$ be a submanifold.

Denote by $\mathcal{T}'$ the restriction of the tangent bundle $\mathcal{T}(M)$ to the submanifold N. This bundle has a metric B' and a connection ∇' induced from Riemannian metric and Levi Civita connection on the bundle $\mathcal{T}(M)$.

Consider the natural imbedding of vector bundles $\mathcal{T}(N) \to \mathcal{T}'$ and endow N with the Riemannian structure given by the form B_N obtained by restriction of the form B'.

Show that the Levi-Civita connection on the Riemannian manifold N is obtained by projection of the connection ∇' on the sub-bundle $\mathcal{T}(N) \subset \mathcal{T}'$ (see problem 5.6).

Remark. This gives one of the ways to investigate Levi-Civita connections. Namely in order to investigate the Levi Civita connection ∇_N on a Riemannian manifold N we can isometrically imbed N into an Euclidean space $M = \mathbf{R}^N$ (maybe of high dimension) and investigate ∇_N as a projection of the standard affine connection ∇_M on the Euclidean space.