

## On the notion of Hilbert polynomial.

Algebraic Geometry and Commutative Algebra

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**I. Preparation about sequences.** Consider the group  $F$  consisting of sequences of rational numbers  $f = \{f(i), i \in \mathbf{Z}\}$ . Let us introduce an equivalence relation on  $F$  by  $f \sim h$  if  $f(i) = h(i)$  for  $i \gg 0$ .

We say that a sequence  $f$  is **eventually polynomial** if there exists a polynomial  $P \in \mathbf{Q}[t]$  such that  $f$  is equivalent to the sequence  $P(i)$ . It is clear that such polynomial  $P$  is uniquely defined.

Consider the difference operator  $\Delta : F \rightarrow F$  defined by  $\Delta(f)(i) = f(i+1) - f(i)$

**1.** Let  $d$  be a natural number. Show that a sequence  $f \in F$  is eventually polynomial of degree  $\leq d$  iff  $\Delta^{d+1}(f) \sim 0$ ; this is also equivalent to the condition that  $\Delta(f)$  is eventually polynomial of degree  $\leq d-1$ .

**II. Hilbert polynomial.** Fix an arbitrary field  $K$ . Consider an algebra  $A = K[x_1, \dots, x_n]$  and introduce on it **algebra filtration**  $\{A_k\}$ , where  $A_k = \{P \in A \mid \deg P \leq k\}$ .

Let  $M$  be a finitely generated  $A$ -module. Fix a system of generators  $m_1, \dots, m_r$  and consider a filtration of  $M$  defined by  $M_k = A_k m_1 + A_k m_2 + \dots + A_k m_r$ .

Our goal is to prove the following fundamental result due to Hilbert.

**Theorem A.** The sequence  $f_M(i) = \dim M_i$  is eventually polynomial (here dimension is over the field  $K$ ).

It is convenient to formulate and prove slightly more general result.

**Definition.** (i) A **filtration** of  $M$  is a collection of finite dimensional subspaces  $M_k \subset M$  defined for all  $k \in \mathbf{Z}$  that satisfies the following conditions.

- (a)  $M_k \subset M_l$  for  $k \leq l$ ,  $M_k = 0$  for  $k \ll 0$  and  $\bigcup M_k = M$ .
- (b)  $A_k M_l \subset M_{k+l}$
- (ii) Filtration  $\{M_k\}$  is called **good filtration** if it satisfies
- (c) For large  $k$  we have  $A_1 M_k = M_{k+1}$ .

Clearly the filtrations considered in Theorem A are good. So we will prove more general result

**Theorem B.** Suppose  $\{M_k\}$  is a good filtration of an  $A$ -module  $M$ .

(i) For any  $A$ -submodule  $L \subset M$  consider the induced filtration on  $L$  defined by  $L_k = L \cap M_k$ . Then it is a good filtration.

(ii) The sequence  $f(i) := \dim M_i$  is eventually polynomial.

Consider the graded algebra  $C = K[t_0, t_1, \dots, t_n]$ . Using the filtration  $\{M_k\}$  on  $M$  construct a graded  $C$ -module  $N = \hat{M} \subset \hat{M} \subset C[t, t^{-1}]$  by  $\hat{M}^k = M_k t^k$ , where  $t_0$  acts as multiplication by  $t$  and  $t_i$  acts as a multiplication by  $t x_i$  for  $i = 1, \dots, n$ .

**2.** Check that a filtration  $\{M_k\}$  is good iff the  $C$ -module  $\hat{M}$  is finitely generated.

For an  $A$ -submodule  $L \subset M$  consider the induced filtration  $\{L_k\}$ . Then  $\hat{L}$  is a  $D$ -submodule of  $D$ -module  $\hat{M}$ . Hence Hilbert basis theorem implies (i).

It is clear that the theorem B follows from the following

**Theorem C.** Consider the algebra  $C = K[t_0, t_1, \dots, t_n]$  and define the grading  $C = \bigoplus C^k$  on it by condition  $\deg(t_i) = 1$ . Fix a graded  $C$ -module  $N = \bigoplus N^k$ .

Suppose we know that  $C$ -module  $N$  is finitely generated. Then the sequence  $f_N(i) := \dim N^i$  is eventually polynomial of degree  $\leq n$ .

**Proof.** Consider the operator  $T : N \rightarrow N$  of degree 1 given by multiplication by  $t_n$ . Let us denote by  $K$  and  $C$  its kernel and cokernel.

**3.** Check that  $f_N(i+1) - f_N(i) \equiv f_C(i+1) - f_K(i)$

Now note that on the modules  $K$  and  $C$  the operator  $t_n$  is zero, so they are finitely generated modules over the algebra  $C' = K[t_0, t_1, \dots, t_{n-1}]$ .

Using induction in  $n$  we can assume that the sequences  $f_K$  and  $f_C$  are eventually polynomial of degree  $\leq n-1$ . But then it means that the sequence  $\Delta(f)$  is eventually polynomial of degree  $\leq n-1$  and hence  $f$  is eventually polynomial of degree  $\leq n$ .

**Remarks.** (i) Note that in fact we start our induction from the case  $n = -1$ , i.e.  $C = K$ .

(ii) The most non-trivial step in this proof is the fact that the  $C$ -module  $K$  is finitely generated - this is Hilbert's basis theorem.

### III. Some problems about Hilbert polynomials.

4. Let  $\mathcal{O}$  be a finitely generated  $K$ -algebra and  $M$  a finitely generated  $\mathcal{O}$ -module.

Let us fix a system of generators  $x_1, \dots, x_n \in \mathcal{O}$ . Then  $M$  becomes a module over the polynomial algebra  $A = K[x_1, \dots, x_n]$ .

Let us choose a good filtration on  $M$  and consider the corresponding Hilbert polynomial  $f_M(i)$ .

(i) Show that the degree  $d(M)$  of the polynomial  $f_M$  and its first coefficient  $e(M)$  do not depend on the choice of a good filtration on  $M$ .

(ii) Show that the degree  $d(M)$  does not depend on the choice of generators of the algebra  $\mathcal{O}$ .

We call this invariant  $d(M)$  the "the functional dimension" of  $M$ .

5. (i) Show that if  $L$  is an  $\mathcal{O}$ -submodule of  $M$  then  $d(M) = \max(d(L), d(M/L))$ .

(ii) Let  $A$  be an endomorphism of an  $\mathcal{O}$ -module  $M$ . Show that if  $A$  is injective then  $d(M/AM)$  is strictly less than  $d(M)$  (we assume  $M \neq 0$ ).

(iii) Suppose that we have a vector space  $M$  that is a module over two commutative finitely generated algebras  $A$  and  $B$ . Let us assume that it is finitely generated over  $A$  and also over  $B$ , so we can define two invariants  $d_A(M)$  and  $d_B(M)$ .

Show that if the actions of  $A$  and  $B$  on the module  $M$  commute, then  $d_A(M) = d_B(M)$ .

6. Let  $X$  be an affine algebraic variety,  $M$  a finitely generated  $\mathcal{O}(X)$ -module. We define the support of  $M$  to be the subset  $\text{supp}(M) \subset X$  defined by the ideal  $I = \text{Ann}(M) \subset \mathcal{O}(X)$ .

Show that  $d(M)$  equals  $\dim \text{supp}(M)$ .

7. Prove that the dimension function  $\dim_H(X)$  defined using Hilbert polynomial definition has the following properties. Let  $\pi : X \rightarrow Y$  be a morphism of affine algebraic varieties

(i) Suppose that  $\pi$  is a finite morphism ( e.g. a closed embedding). Then  $\dim_H X \leq \dim_H Y$ .

(ii) Suppose that  $\pi$  is a finite epimorphism. Then  $\dim_H X = \dim_H Y$ .

(iii) Suppose  $\pi$  is an imbedding of a basic open subset (i.e.  $X = Y_f$ ). Then  $\dim_H X \leq \dim_H Y$

8. Show that Hilbert polynomial definition of dimension for algebraic varieties is equivalent to Krull's definition.

(\*) 9. Using Hilbert polynomial definition of dimension prove directly the **Principle ideal theorem**.

Let  $X$  be an irreducible affine algebraic variety,  $f \in \mathcal{O}(X)$ ,  $Z = Z(f)$  the zero set of the function  $f$ . Suppose that  $\dim Z \leq \dim X - 2$ . Then  $Z$  is empty.