Final take-home exam.

Algebraic Geometry and Commutative Algebra

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This is a Take-Home exam. You should try to solve as many problem as you can. If you have some ideas how to solve the problem but do not have complete solution - write these ideas.

It is enough to solve correctly 9 problems out of 12 in order to get 90 points.

You should send me the solutions of the exam not later than September 10, 2011 via email (as a scan or a TEX file).

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Good Luck

1. Consider the hypersurface $X \subset \mathbf{A}^3$ defined by equation $x^3 + y^5 + x^7 = 0$.

(i) Describe all singular points of X.

(ii) Let Y be the closure of X in $\mathbf{P}^3$. Describe all the singular pints of Y (you may assume that $char(k) = 0$).

2. Let V be a finite-dimensional vector space over k; we will also denote by V the corresponding algebraic variety.

Consider the natural projection $p : V^* = V \setminus 0 \to \mathbf{P}(V)$.

Show that if $\dim V > 1$ then this morphism can not be extended to a regular morphism $p : V \to \mathbf{P}(V)$.

3. Consider quadratic cone $V \subset \mathbf{A}^3$ defined by equation $xy - z^2 = 0$.

(i) Compute $\dim V$. Describe the sheaf of differentials Ω_V.

(ii) Describe Zariski tangent spaces at all points of V.

(iii) Find a closed irreducible subvariety $W \subset V$ of codimension 1 such that the ideal $I_W \subset \mathcal{O}(V)$ is not a principal ideal (and prove that it is not principal).

4. Let $\nu : F \to H$ be a morphism of coherent sheaves on an algebraic variety X. For every point $x \in X$ we denote by $\nu_x : F|_x \to H|_x$ the corresponding morphism of fibers.

(i) Assume that at some point $a \in X$ the morphism ν_a is an epimorphism. Show that there exists an open neighborhood U of the point a in X such that the restriction of the morphism ν to U is an epimorphism of sheaves.

(ii) Give an example of morphism ν such that at some point $a \in X$ the morphism ν_a is injective, but ν is not injective in any neighborhood U of a.

5. Let $\nu : F \to H$ be a morphism of coherent sheaves on an algebraic variety X as in problem 4. Assume that $\mathcal{F}$ and $\mathcal{H}$ are locally free $\mathcal{O}$-modules of rank 4.

Suppose we know that at some point $a \in X$ the morphism ν_a has rank ≤ 2. Show that if X is irreducible of dimension 5 then there are other points with this property.

6. Let $X \subset \mathbf{P}(V)$ be an irreducible smooth hypersurface. Consider the natural morphism $\nu : X \to \mathbf{P}(V^*)$ that assigns to every point $a \in X$ the hyperplane $\Pi_a \subset \mathbf{P}(V)$ that is tangent to X at point a. We say that a point $a \in X$ is **regular** if the differential $d\nu$ of the morphism ν at this point is an embedding.

We say that the hypersurface X is regular if it has at least one regular point.

(i) Choose affine coordinates in a neighborhood of a point a and write X by an equation $f = 0$ in this neighborhood. In terms of function f describe when the point a is a regular point of X.

(ii) Let $X_1, X_2, ..., X_n$ be regular hypersurfaces in $\mathbf{P}^n$. Show that there exists a hyperplane Π that is tangent to all of them.

7. Let $\pi : X \to S$ be a morphism of algebraic varieties. We consider it as a family of algebraic varieties $X_s := \pi^{-1}(s)$ parameterized by points s of the base S.

(i) Let $S_{-\infty}$ be the set of points $s \in S$ such that the variety X_s is empty. Show that $S_{-\infty}$ is a constructible subset of S.

(ii) For every i denote by S_i the set of points $s \in S$ such that $\dim X_s \leq i$. Show that S_i is a constructible subset of S.

8. Suppose that the morphism $\pi : X \to S$ in problem 7 is a projective morphism (e.g. X is a closed subset of $S \times \mathbf{P}^N$ and p is its projection to S).

Show that in this case the subsets $S_i \subset S$ are open.

9. Let A be a commutative algebra, M a finitely generated A-module and T an endomorphism of the A-module M.

Show that T is epimorphic if and only if it is an isomorphism.

10. Let X be a hypersurface of a projective space $\mathbf{P}(V)$.

(i) Show that X can be defined by a homogeneous polynomial $f \in \mathcal{O}(V)$ such that f generates a radical ideal in $\mathcal{O}(V)$.

(ii) Show that the set X_{sing} of singular points of X can be described by an equation $df = 0$.

11. Consider the Serge morphism $\nu : \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N$, where $N = mn + m + n$. Show that this is a closed embedding. Compute the degree of its image $\Sigma_{m,n} \subset \mathbf{P}^N$.

Hint. Describe the invertible $\mathcal{O}$-module corresponding to the morphism ν.

12. Let $\pi : X \to Y$ be a morphism of smooth algebraic varieties. For every point $x \in X$ we denote by $D_x\pi : T_xX \to T_{\pi(x)}Y$ its differential at the point x.

Consider subsets $I, S \subset X$ defined by

$I = \{x \in X | \ D_x\pi$ is a monomorphism $\}$
$S = \{x \in X | \ D_x\pi$ is an epimorphism $\}$.

Show that I and S are open subsets of X.