Problem assignment 3.

Algebraic Geometry and Commutative Algebra

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[P] 1. (LA) Let V be a vector space over an algebraically closed field k, $T : V \to V$ a linear operator. By definition the **spectrum** $Spectr(T)$ of the operator T is a subset of numbers $\lambda \in k$ such that the operator $T - \lambda$ is not invertible.

(i) Suppose T is locally nilpotent, i.e. for any vector $v \in V$ we have $T^n v = 0$ for large n. Show that then $Spectr(T) = \{0\}$.

(ii) Show that if the space $V \neq 0$ is countable dimensional and the field k is algebraically closed and uncountable, then conversely the condition $Spectr(T) = \{0\}$ implies that T is locally nilpotent.

The following definition will play an important role in what follows.

(CA) **Definition**. Let B be commutative ring. A (commutative) B**-algebra** is a commutative ring A equipped with a homomorphism of rings $\nu : B \to A$. This naturally defines on A a structure of a B-module. B-algebra A is called **finite** if A is finitely generated as B-module (equivalent terminology - "A is finite over B" or "morphism $\nu : B \to A$ is finite").

(CA) **2.** (i) Consider homomorphisms of rings $C \to B \to A$. Show that if A is finite over B and B is finite over C then A is finite over C.

(ii) Choose a monic polynomial $P \in B[t]$ and consider the B-algebra $A = B[t]/PB[t]$. Show that A is finite over B.

We fix an infinite field K and consider the polynomial algebra $\mathcal{P}_n = K[x_1, ..., x_n]$.

[P] 3. Prove **Noether's Normalization Lemma.** Let M be a non-zero finitely generated $\mathcal{P}_n$-module. Then there exist a number d and a system of linear forms $y_i = \sum a_{ij} x_j,\ 1 \leq i \leq d$ such that

(α) module M is finitely generated over the algebra $B = K[y_1, ..., y_d]$

(β) Annihilator of M in B is 0.

In fact given one such system show that almost every system of forms y_i has these properties.

∇ **Remark.** Try to formulate and prove the analogue of Noether's Normalization Lemma for the case of finite field K.

[P] 4. Fix an infinite field K and consider the algebra $\mathcal{P}_n = K[x_1, ..., x_n]$. Let M be a non-zero finitely generated $\mathcal{P}_n$-module. Prove the following Nullstellensatz property.

(i) If $n > 0$ then there exists a non-zero linear form $y = \sum a_j x_j$ that satisfies the following condition:

(*) There exists a monic polynomial $P \in K[t]$ such that $P(y)M \neq M$.

(ii) There exists an ideal $J \subset \mathcal{P}_n$ such that the quotient $N = M/JM$ is a non-zero module that has finite dimension over the field K.

(iii) Prove that any non-zero linear form y satisfies the property (*) above.

(iv) Show that if $\mathfrak{m} \subset \mathcal{P}_n$ is a maximal ideal then the quotient field $L = \mathcal{P}_n/\mathfrak{m}$ is a finite extension of the field K.

5. Here we present an elementary proof of Nakayama lemma.

Let J be an ideal in a commutative ring A. We set $R = 1 + J \subset A$. Clearly $R \cdot R \subset R$ and $R + J \subset R$.

Lemma (Nakayama). Let M be a finitely generated A-module such that $JM = M$. Then there exists an element $r \in R$ such that $rM = 0$.

Remark. Informally it means that if there exists a procedure to write every element $m \in M$ as a sum $\sum j_i m_i$ then there exists an element $j \in J$ that gives a universal such procedure, namely $m \equiv jm$.

In particular this implies that for any A-submodule $L \subset M$ we have $JL = L$.

Proof. We use induction in number n of generators. Let x be one of generators of M and denote by $L = Ax \subset M$ the submodule generated by x. Using the induction assumption for the module M/L we can find an element $r' \in R$ such that $r'M \subset L$.

This implies that $r'x \in r'JM = Jr'M \subset JL = Jx$. Thus we can find an element $j \in J$ such that $r'x = jx$.

This shows that the element $r'' = r' - j$ annihilates x (and hence L). Since $r'M$ lies in L we see that the element $r = r'' \cdot r'$ annihilates M.

[P] 6. Let T be an endomorphism of a finitely generated A-module M. Suppose we know that T is epimorphic. Show that T is invertible. Moreover show that it is invertible on any T-invariant A-submodule $L \subset M$.

7. Let $\nu : B \to A$ be a finite morphism of k-algebras. Show that the corresponding map of sets $\nu^* : M(A) \to M(B)$ is closed and has finite fibers.

(CA) Let us introduce some notions in the theory of schemes that we will discuss later in more detail. For any finitely generated k-algebra A we defined a set $M(A) := Mor_{k-alg}(A, k)$. As follows from Nullstellensatz this set also can be interpreted as the set of all maximal ideals of A. Grothendieck generalized this notion as follows.

Definition. Let A be an arbitrary commutative ring with 1. We define the set $SpecA$ to be the set of **prime** ideals $\mathfrak{p} \in A$. Let us remind that **prime ideal** is an ideal $\mathfrak{p} \subset A$ such that $A/\mathfrak{p}$ is a non-zero algebra without zero divisors.

∇ (CA) **8.** (i) Show that if $A \neq 0$ then $SpecA$ is non-empty.

(ii) Show that the intersection of all prime ideals of A is the nillradical of A (i.e. the set of all nilpotent elements in A).

(iii) Define Zariski topology on $SpecA$. Show that every morphism of algebras $\nu : B \to A$ induces a continuous map $\nu^* : SpecA \to SpecB$.

(iv) Show that if the morphism ν is finite then the map ν^* is closed.

Remark. If we assign to every ring A the set $Specm(A)$ of its maximal ideals then in some cases we will not be able to construct a map $\nu^* : SpecmA \to SpecmB$. Give an example.

9. Prove **Hamilton - Cayley Identity (HC)**.

Let A be a commutative algebra. Consider an $n \times n$ matrix X with entries in A, $X = (a_{ij}) \in Mat(n, A)$. Define the characteristic polynomial $P_X \in A[t]$ of the matrix X by formula $P_X = \det(t - X)$. Prove the identity

(**HC**) The matrix $P_X(X)$ equals 0.

Sketch. This is an identity of some polynomials in coefficients a_{ij} of matrix X. Show that it is enough to prove (HC) for the ring $B = \mathbf{Z}[x_{ij}]$ and a specific matrix X with entries $a_{ij} = x_{ij}$. Show that we can imbed the ring B into the field $\mathbf{C}$ of complex numbers, so it is enough to prove (HC) for the case $A = \mathbf{C}$. Show that (HC) holds on the subset $Mat_n^{reg}(\mathbf{C}) \subset Mat_n(\mathbf{C})$ of matrices with distinct eigenvalues. Show that this subset is given by inequality $Disc(X) \neq 0$ for some non-zero polynomial function $Disc$ on $Mat(n, \mathbf{C})$.

[P] 10. Show that (HC) implies Nakayama lemma. Namely show that

(i) Given a finitely generated A-module M and its endomorphism X one can find a monic polynomial $P \in A[t]$ such that $P(X) = 0$.

(ii) Moreover, if we know that the image $X(M)$ lies in the subset JM for some ideal $J \subset A$ then we can choose the polynomial P to be of the form $P = t^n + a_1 t^{n-1} + ... + a_n$, where all coefficients a_i lie in J. (**Hint.** First prove this for endomorphisms of a free module F and then lift the endomorphism X to an endomorphism of a free module F covering M).