Algebraic Geometry and Commutative Algebra

Problem assignment 4.

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Definition. Let C be a category. An object $F \in Ob(C)$ is called **a final object** if for any object $X \in Ob(C)$ the set $Mor_C(X, F)$ consists of exactly one element.

Similarly one defines the notion of **an initial object**.

1. (i) Show that if there exists a final object F in C then it is defined uniquely up to unique isomorphism.

(ii) Let C be a category and R an object of C. We define a new category C_R - category of objects over R - as follows

Object of C_R is a pair (X, p), where $X \in Ob(C)$ and $p : X \to R$.

Morphism $\nu : (X, p) \to (Y, q)$ in the category C_R is a morphisms $\nu : X \to Y$ in C that makes the diagram commute, i.e. $q \cdot \nu = p$.

Show that the category C_R always has a final object.

(iii) Let R, S be objects of C. Define in the natural way the category C_{RS}.

If this category has a final object (Z, p_R, p_S) we call the object $Z \in Ob(C)$ the product of objects R, S and the morphisms p_R, p_S projections.

As follows from (i) this product is defined uniquely up to canonical isomorphism. Check for yourself that this definition coincides with the one discussed in class.

(iv) Define the notion of sum of objects in a category.

2. In any category C define the notions of "monomorphism", "epimorphism", "isomorphisms". Give example when "monomorphism + epimorphism does not imply isomorphism.

3. Fix a ring A and consider the category $\mathcal{M}(A)$ of A-modules.

(i) Show that this category has arbitrary (e.g. infinite) sums.

(ii) Let M be a finitely generated A-module. Show that the functor $X \mapsto Hom(M, X)$ from the category $\mathcal{M}(A)$ to the category Ab of abelian groups sends direct sums to direct sums.

The converse statement is not always true. Later we will discuss some strengthening of it that is correct. The idea is that the notion "finitely generated" can be expressed in purely categorical terms.

(iii) Show that an extension of two finitely generated modules is finitely generated.

Definition. An A-module M is called **Noetherian** if any A-submodule $L \subset M$ is finitely generated.

A ring A is called Noetherian if any finitely generated A-module M is Noetherian.

[P] 4. (i) Show that an A-module M is Noetherian iff it satisfies the following condition

(*) Any increasing chain of submodules $L_1 \subset L_2 \subset ... \subset M$ is stable, i.e. modules L_i coincide for large i-s.

(ii) Let M be an extension of L and N. Show that M is Noetherian iff L and N are Noetherian.

(iii) Show that a ring A is Noetherian iff A is Noetherian as A-module.

5. (i) Let $A = k[t_1, t_2, ...]$ be the algebra of polynomials in infinite number of generators. Show that A is not Noetherian

(ii) Let $A = k[x, y]$. Consider a subalgebra $B = k + xA \subset A$. Show that the algebra B is not Noetherian (and hence not finitely generated as k-algebra).

Definition. Let X be a topological space.

(i) The space X is called **irreducible** if it is not empty and it can not be written as a union of two proper closed subsets $F_1, F_2 \subsetneq X$

A subset $Z \subset X$ is called **irreducible** if it is irreducible in induced topology.

(ii) The space X is called **Noetherian** if any increasing sequence of open subsets $U_1 \subset U_2 \subset ...$ in X is stable.

[P] 6. (i) Show that a non-empty space X is irreducible iff it satisfies the following condition:

(*) Every non-empty open subset $U \subset X$ is dense in X.

(ii) Show that a subset $Z \subset X$ is irreducible iff its closure $cl(Z)$ is irreducible.

(iii) Let $\nu : X \to Y$ be a continuous map of topological spaces. Show that if a subset $Z \subset X$ is irreducible then its image $\nu(Z)$ is irreducible.

Definition. Let X be a Noetherian space. Define **dimension** of X to be supremum of lengths d of strict chains of irreducible closed subsets $F_0 \subsetneqq F_1 \subsetneqq ... \subsetneqq F_d \subset X$.

[P] 7. Let X be a Noetherian space.

(i) Show that for any open subset $U \subset X$ we have $\dim U \leq \dim X$. Show that if X is a union of open subsets U_α then $\dim X = \max \dim U_\alpha$.

(ii) Suppose X is irreducible and $F \subsetneqq X$ is a closed subset. Show that $\dim X \geq \dim F + 1$.

[P] 8. Let X be an algebraic variety.

(i) Show that $\dim X$ equals to the maximum of dimensions of irreducible components of X. More generally, if X is presented as a union of finite collection of locally closed subsets Y_i then $\dim X = \max \dim Y_i$.

(ii) Let Y be a locally closed subset of X. Denote by $cl(Y)$ the closure of Y in X and define the **boundary** ∂Y of Y by $\partial Y = cl(Y) \setminus Y$.

Show that $\dim cl(Y) = \dim Y$ and $\dim \partial Y \leq \dim Y - 1$.

Definition. If X is an algebraic variety and $a \in X$ we define the local dimension $\dim_a X$ to be the minimum of $\dim U$ for all open neighborhoods U of a.

[P] 9. Suppose X is irreducible. Show that for any point $a \in X$ the local dimension $\dim_a X$ equals $\dim X$.

Remark. Analogous statement does not hold for arbitrary Noetherian spaces.

[P] 10. Let X be an affine irreducible variety. Denote by $k(X)$ the field of rational functions on X. Show that the dimension $\dim X$ equals to the transcendence degree of the field $k(X)$ over k.