Problem assignment 3.

Functions of Complex variables, II

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Let U be an open subset in $\mathbf{R}^2$. We denote by $S(U)$ the algebra of smooth (i.e. C^∞) complex valued functions on U. We sometimes write a function $f \in S(X)$ as $f = u + iv$, where u, v are real valued functions. We denote by $\bar{f}$ the complex conjugate function $u - iv$.

Choose coordinate system (x, y) on $\mathbf{R}^2$ and denote by $\partial_x, \partial_y : S(U) \to S(U)$ the operators of partial derivatives.

For every function $f \in S(U)$ we denote by the same symbol f the operator $f : S(U) \to S(U)$ given by multiplication with f.

Definition. (i) The algebra of differential operators on U is the algebra of endomorphisms of the space $S(U)$ generated by multiplication operators f for $f \in S(U)$ and operators ∂_x, ∂_y. We denote this algebra by $D(U)$.

(ii) The space of derivations $Der(S(U)) \subset D(U)$ of the algebra $S(U)$ is defined as $Der(S(U)) := S(U)\partial_x \oplus S(U)\partial_y$.

1. (i) Check the following formulas

$[\partial_x, f] = \partial_x(f), [\partial_y, f] = \partial_y(f), [\partial_x, \partial_y] = 0$

(ii) Show that every differential operator D can be uniquely written as $D = \sum_{m,n} f_{mn} \partial_x^m \partial_y^n$.

Remark. The space $Der(S(U))$ has a structure of a Lie algebra with respect to operation $[\ ,\ ]$. Any derivation d satisfies the Leibnitz rule $d(fh) = dfh + fdh$; in fact this property characterizes all derivations and hence gives the coordinate free definition of the space $Der(S(U))$. One can also define the algebra $D(U)$ directly, without using any coordinate system.

Now consider a complex function $z = x + iy$ and its conjugate function $\bar{z}$. It turns out that in some formal algebraic sense one can consider the pair of functions $z, \bar{z} \in S(U)$ as a kind of " coordinate system" on U. As we will see this gives a very convenient formalism for studying functions on complex plane.

Define the operator ∂_z as a unique derivation of the algebra $S(U)$ that sends z to 1 and $\bar{z}$ to 0. Similarly define the operator $\partial_{\bar{z}}$.

[P] 2. Show that $\partial_z = 1/2(\partial_x - i\partial_y), \partial_{\bar{z}} = 1/2(\partial_x + i\partial_y)$. In particular $\partial_{\bar{z}}\bar{f} = \overline{\partial_z f}$.

Show that these operators commute. Show that any differential operator D can be uniquely written as $D = \sum_{m,n} f_{mn} \partial_z^m \partial_{\bar{z}}^n$.

This shows that in algebraic manipulations these operators behave in exactly the same way as partial derivatives of a coordinate system.

3. (**Cauchy - Riemann equation**). Let $f \in C^1(U)$. Show that f is holomorphic iff $\partial_{\bar{z}}(f) = 0$. In this case $\partial_z f$ coincides with the derivative f' of the holomorphic function f.

Definition. We define the **Laplace operator** Δ by formula $\Delta = \partial_x^2 + \partial_y^2 = 4\ \partial_z \cdot \partial_{\bar{z}}$.

A function $h \in C^2(U)$ is called **harmonic** if $\Delta h = 0$.

[P] 4. (i) Let $f = u + iv$ be a holomorphic function. Show that its real and imaginary parts u, v are harmonic. Show that $\partial_z u = 1/2\ \partial_z f = 1/2\ f'$.

(ii) Show that if the domain U is connected and simply connected then for any real harmonic function h on U there exists a holomorphic function f on U such that h is the real part of f. Such function f is defined uniquely up to addition of an imaginary constant function.

[P] 5. (ii) Let f be a holomorphic function on U and H be a harmonic function on the plane. Show that the function $H(f(z))$ is harmonic.

(ii) Let $f(z)$ be a holomorphic function without zeroes. Show that the function $h(z) = \ln(|f(z)|)$ is harmonic.

[P] 6. (i) Let $f = \sum a_n z^n$ be a holomorphic function on the unit disc D continuous on the boundary $S^1 = \partial D$. Show that we can compute the Taylor coefficients of f in terms of their boundary values on S^1.

$a_n = \frac{1}{2\pi}\int_{S^1} f(\xi)\xi^{-n} d\theta = Av_{S^1} f(\xi)\xi^{-n}$ (Av is the notation for the average of a function).

(ii) Show that the last formula holds for disc of any radius and deduce that an entire function f that is bounded by some polynomial $Q(x, y)$ is a polynomial function in z.

Schwartz reflection principle. Let U, V be domains and $\tau : U \to V$ an anti-holomorphic diffeomorphism. For a holomorphic function f on V we define a function f^τ on U by formula $f^\tau(z) = \overline{f(\tau(z))}$. Show that the function f^τ is holomorphic.

In interesting cases the map τ^2 is the identity map and then the set γ of fixed points of τ is some curve γ. In this case on curve γ we have the identity $f^\tau(z) = \overline{f(z)}$. Two important examples of such maps τ are:

(i) $\tau(z) = \bar{z}$; in this case $\gamma = \mathbf{R} \subset \mathbf{C}$

(ii) $\tau(z) = \overline{z^{-1}}$: in this case $\gamma = S^1$ - unit circle.

[P] 7. (i) Let $f = u + iv$ be a holomorphic function on the unit disc D continuous on the boundary $S^1 = \partial D$. Let us assume that $f(0) = 0$.

Show that $f(z) = \frac{1}{2\pi i}\int_{S^1}(f(\xi) + \bar{f}(\xi))\frac{d\xi}{\xi - z} = \frac{1}{\pi i}\int_{S^1} u(\xi)\frac{d\xi}{\xi - z} = \frac{1}{2\pi}\int_{S^1} u(\theta)\frac{\xi + z}{\xi - z} d\theta$

(ii) Deduce from this Poisson formula that computes the values of any harmonic function h on the unit disc D in terms of its boundary values on $S^1 = \partial D$.

8. (Borel - Carathéodory Theorem).

(i) In problem 7 assume that $f(0) = 0$ and $u(z) \leq 1$ for all z. Show that then we have a bound $|f(z)| \leq \frac{2|z|}{1-|z|}$.

Hint. Consider holomorphic function $h(z) = f(z)/(2 - f(z))$. Using Schwarz lemma show that $|h(z)| \leq |z|$.

(ii) Using this inequality show that if for an entire function f its real part u is bounded by a polynomial function $Q(x, y)$ then f is a polynomial function in z.

Remark. If we do not assume in (i) that $f(0) = 0$ we still can get an estimate

$|f(z)| \leq \frac{2|z|}{1-|z|} + \frac{1+|z|}{1-|z|}|f(0)|$