Problem assignment 5.

Functions of Complex variables, II

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1. Compute the sum of residues of the function $f(z) = \frac{(z^2+3z+1)(z^3-2)}{z^4-5z^2+z+1}$ at all points $z \in \mathbf{C}$.

2. (i) Evaluate the integral $\int_{-\infty}^{\infty} f(x)dx$, where $f(x) = \sin x/x$. Evaluate the integral $\int_0^{\infty} f(x)dx$.

Hint. Define the **principle value** integral $PI \int h(x)dx$ as an average of two integrals $\int_{z=x\pm i\epsilon} h(z)dz$. Compute this PV integral for functions $h(z) = \exp(\pm iz)/z$.

(ii) Evaluate the integral $\int_0^{\infty} (\sin x/x)^3 \ dx$

3. Evaluate the integrals

(i) $\int_0^{2\pi} (a + \cos x)^{-1} dx$

(ii) $\int_{|z|=5} \frac{zdz}{\sin z(1-\cos z)}$

(iii) $\int_{-\infty}^{0} Q(x)dx$, where $Q(x) = \frac{x}{(x-1)(x-3)(x-7)}$.

Hint. Compute in two ways the integral $\int_C \log z \ Q(z)dz$, where C is a contour around the half-axis of negative real numbers.

4. Let f be an entire function. Suppose we know that $Re(f(z)) \leq C(1 + |z|^{10})$. Show that f is a polynomial function.

5. Show that the function $(\sin z)^{-1}$ is given by an infinite series $1/z + \sum'_{n\geq 1} \frac{(-1)^n}{z^2-\pi n^2}$.

6. Evaluate the product $\prod_{n\geq 1}(1 - \frac{z^3}{n^3})$ in terms of Γ-function.

7. 1. Let L be a lattice in $\mathbf{R}^n$ (this means that L is a discrete subgroup of $\mathbf{R}^n$ and the quotient space $T = \mathbf{R}^n/L$ is compact.

(i) Show that there exists a basis $(e_1, ..., e_n)$ of $\mathbf{R}^n$ such that $L = \sum_i \mathbf{Z}e_i$.

(ii) Let f be a function on L that for some real number α satisfies the bound $|f(x)| \leq C|x|^{\alpha}$ when $|x|$ is large.

Show that if $\alpha < -n$ then the series $\sum_{x\in L} f(x)$ is absolutely convergent.

Hint. Compare this sum with the integral $\int_{|x|>1} |x|^{\alpha} dx$.

Remark. Show that this sum is absolutely convergent if f satisfies a weaker bound $|f(x)| \leq C|x|^{-n}(\log|x|)^{\beta}$ for large $|x|$ provided $\beta < -1$.

8. Fix a lattice $L \subset \mathbf{C}$ and denote by $\mathcal{P}(z)$ the Weierstrass function and by $\mathcal{P}'(z)$ its derivative. Show that the function $f(z) = \mathcal{P}(2z)\mathcal{P}'(z)^2$ can be expressed as a polynomial in function $\mathcal{P}(z)$.

9. Consider a non-linear ordinary differential equation for a function f

(*) $f'(x)^2 = 4f(x)^3 + af(x) + b$.

Can you write in some way general solution of this equation.