Final Exam

Algebraic Geometry and Commutative Algebra II Fall 2011

Joseph Bernstein February 15, 2012.

This is the final take-home exam. You should hand in your solutions before March 10. You can put the solutions into my mailbox. You might also send the solutions to me via email - as a tex file or as a scan.

You are advised to think about all problems. The complete solution of any 8 out of 12 problems guarantees you 90 points.

A have included also 3 additional problems A1, A2, A3 about Weierstrass points on algebraic curves. They are not part of the exam, but I encourage you to think about them after you finish writing the exam.

In general, if you have some ideas about a problem but are not sure you can give a complete solution, please write down the ideas you have.

GOOD LUCK

In this exam we mostly deal with algebraic varieties over an algebraically closed field K.

1. Consider a family F_s of coherent $\mathcal{O}$-modules on the projective space $X = \mathbf{P}^n$ parameterized by points s of an algebraic variety S. Formulate precisely what does it mean.

Show that there exists k such that for all $s \in S$ the $\mathcal{O}_X$-module $F_s(k)$ is acyclic and is generated by global sections.

2. Let F be a coherent $\mathcal{O}$-module on projective space $X = \mathbf{P}^n$. Fix a number l and consider the sequence $b(k) := \dim H^l(X, F(k))$. Show that there exists a constant C such that $b(k) \leq C|k|^l$ when $k \to -\infty$.

3. Let X be an algebraic variety. Consider a function $u(x)$ on X defined by $u(x) = \dim T_x(X)$ - dimension of the Zariski tangent space. Show that this function is upper semicontinuous. Describe all points of local minimum of this function.

4. Let $\pi : X \to Y$ be a finite morphism of algebraic varieties. Suppose Y is projective and L is an ample invertible $\mathcal{O}_Y$-module. Show that then X is also projective and $\mathcal{O}_X$-module $N = \pi^*(L)$ is ample.

5. Let $\pi : C \to E$ be a dominant morphism of smooth projective curves, $d = deg(\pi)$.

(i) Show that genus of C is greater or equal to the genus of E.

Give examples of morphisms of degree $d > 1$ for which genus of C equals genus of E. Can this happen if genus is bigger than 1 ?

(ii) Suppose that π is unramified everywhere. Compute the genus of C if you know that the genus of E equals g'.

More generally, suppose that π is unramified except points $a_1, ..., a_r \in C$ at which it has ramification indexes $m_1, ..., m_r$ (define these indexes). In this case write a formula for the genus of C.

(iii) Show that projective line does not have unramified coverings.

(iv) Prove the following

Statement (Lüroth's Theorem). Let L be any K-subfield of the field $K(t)$ of rational functions in one variable. If $L \neq K$, then L is isomorphic to the field $K(t)$.

6. By definition the **Picard group** $Pic(X)$ of an algebraic variety X is the group of isomorphism classes of invertible $\mathcal{O}_X$-modules. Compute Picard groups of the following algebraic varieties:

(i) $X = \mathbf{A}^1$ (ii) $X = \mathbf{A}^1 \setminus 0$

(iii) $X = \mathbf{A}^2$ (iv) $X = \mathbf{P}^1$

(v) X is a non-separated variety obtained from $\mathbf{A}^1$ by doubling point 0.

(vi) X is a projective curve $X \subset \mathbf{P}^2$ defined by homogeneous equation $x^3 - y^2z + xyz = 0$

Hint. Show that X is a rational curve with a node.

7. Show that for any connected variety S the natural morphism $Pic(S) \times Pic(\mathbf{P}^1) \to Pic(S \times \mathbf{P}^1)$ is an isomorphism.

8. Let X be a smooth hypersurface of degree 4 in $\mathbf{P}^3$. Let K be its canonical bundle.

(i) Show that the bundle K is isomorphic to $\mathcal{O}_X$.

(ii) Compute dimensions of the spaces $H^i(X, \mathcal{O}_X)$.

9. Let X, Y be smooth hypersurfaces in $\mathbf{P}^3$ defined by homogeneous polynomials f, h of degrees 5 and 7. Let us assume that they intersect transversally. Consider their intersection $C = X \bigcap Y$. What can be the genus of the curve C ?

10. Let S be a connected, smooth projective surface, H an ample divisor on S. For any two divisors D, E on S we denote by $B(D, E)$ their intersection index.

(i) Show that for any effective divisor D on S we have $B(H, D) \geq 0$ and if $D \neq 0$ then $B(H, D) > 0$.

(ii) Show that for any divisor D we have $B(D, D) \cdot B(H, H) \leq B(H, D)^2$.

(iii) Show that if the ample divisor H is an effective divisor then it is geometrically connected.

In other words this means that any hyperplane section of the surface $S \subset \mathbf{P}^N$ is connected.

11. Let $L/\mathbf{Q}$ be a finite field extension. Denote by D the set of elements of L that are integral over $\mathbf{Z}$.

(i) Show that D is a Noetherian ring finite over $\mathbf{Z}$.

(ii) Show that for any maximal ideal $\mathfrak{m} \subset D$ the local algebra $A = D_{\mathfrak{m}}$ is a discrete valuation ring.

(iii) Show that the cohomological dimension of the ring D equals 1.

12. Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and $k = A/\mathfrak{m}$ its quotient field.

Suppose we know that $Ext^i_A(k, k) = 0$ for $i > N$. Show that the cohomological dimension of the algebra A is bounded by N.

Note. In fact in order to show that the cohomological dimension of A is bounded by N it is enough to check that $Ext^i_A(k, k) = 0$ for $i = N + 1$. Can you prove this ?

Additional Problems.

A1. Let P be a point of a curve C. A natural number n is called a **Weierstrass gap** at P if $l(nP) = l((n-1)P)$.

(i) Show that there are exactly g gaps at the point P.

(ii) Suppose that 2 is not a gap at P. In this case find all the gaps at P. Show that X is a hyperelliptic curve, i.e. there exists a double cover $\tau : C \to \mathbf{P}^1$.

A2. Let C be a smooth projective curve. A point $P \in C$ is called a **Weierstrass point** if the sequence of gaps n_i is different from $1, 2, ..., g$. In other words this means that the Weierstrass number $w(P) = \sum n_i - g(g+1)/2$ is positive.

Show that the following statements are equivalent:

(i) P is a Weierstrass point of C.

(ii) $l(gP) > 1$

(iii) $\Gamma(C, \Omega(-gP)) \neq 0$

(iv) The quotient morphism $r : \Gamma(C, \Omega) \to \Gamma(C, \Omega/\mathfrak{m}_P^g\Omega)$ is not an isomorphism.

A3. (*) (i) Show that the sum of Weierstrass numbers $\sum w(P)$ over all points $P \in C$ equals $g(g^2 - 1)$.

(Hint. Show that the highest exterior power of the morphism r in problem A2(iv) defines a section of the sheaf $\Omega^{\otimes k}$, where k = g(g+1)/2).

(ii) Show that the set of Weierstrass points is finite and is nonempty if $g > 1$.

(iii) Suppose that the genus $g > 1$. Show that in this case the group of automorphisms of the curve C is finite.

Show that if $g \leq 1$ then the group of automorphisms of the curve C is infinite.