Problem assignment 11.

Algebraic Geometry and Commutative Algebra

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In what follows we fix a smooth projective curve C.

First let us describe a more geometric approach that describes the group $Div(C)/PrinDiv(C)$.

Consider the category $\mathcal{P}ic(C)$ whose objects are invertible $\mathcal{O}$-modules L. We denote by $Pic(C)$ the set of isomorphism classes in $\mathcal{P}ic(C)$. The set $Pic(C)$ has a natural structure of an abelian group defined by $[L] \cdot [N] = [L \otimes N]$.

To any divisor D on C we assign an invertible $\mathcal{O}$-module $\mathcal{O}(D)$ defined as follows:

$\Gamma(U, \mathcal{O}(D)) = \{f \in k(C)^* \mid div(f) + D \geq 0 \text{ on } U\} \bigcup 0$.

1. Show that the morphism $D \mapsto [\mathcal{O}(D)]$ defines an isomorphism of abelian groups $Div(C)/PrinDiv(C) \simeq Pic(C)$.

Hint. Consider the category $\mathcal{P}ic'(C)$ consisting of pairs (L, ξ) where L is an invertible $\mathcal{O}$-module and ξ a non-zero rational section of L. Show that the correspondence $D \mapsto (\mathcal{O}(D), 1)$ defines an isomorphism of the set $Div(C)$ with the set of isomorphism classes of the category $\mathcal{P}ic'(C)$. Also show that the category $\mathcal{P}ic'(C)$ is discrete (e.i. its objects do not have non identity automorphisms).

Remark. Given a divisor D we define an autofunctor - "twist by D"- on the category $\mathcal{P}ic'(C)$ by $L \mapsto L(D) = \mathcal{O}(D) \otimes L$. In other words

$\Gamma(U, L(D)) := \{$rational secions ξ of L that satisfy the following condition $div_L(\xi) + D \geq 0\}$.

Definition. For an invertible $\mathcal{O}$-module L we define the following invariant $l(L)$: we consider the space of global sections $\Gamma(L) := \Gamma(C, L)$ and set $l(L) := \dim \Gamma(L)$.

2. Define the degree $deg(L)$ and show that if $deg(L) < 0$ then $l(L) = 0$ and if $deg(L) \geq 0$ then $l(L) \leq deg(L) + 1$.

Definition. We define another important invariant $h(L)$ of the invertible $\mathcal{O}$-module L using formula $l(L) - h(L) \equiv deg(L) + 1 - g$.

3. Show that for all L the number $h(L)$ is non-negative and when $deg(L) > 2g - 2$ we have $h(L) = 0$.

Our next goal is to give a "geometric" construction of the invariant h. Namely we will construct a functor $H : \mathcal{P}ic(C) \to Vect$ such that $h(L) = \dim H(L)$. We will see later that this functor coincides with the cohomology functor $L \mapsto H^1(C, L)$. Compare that $l(L) = \dim \Gamma(L)$, where $\Gamma(L) = H^0(C, L)$.

Let T be an effective divisor. We will be interested in the case when $deg(T) \gg 0$. Also for simplicity we assume that T is simple. i.e. all coefficients n_a are 0 and 1. Thus we can consider T as a finite subset of C.

For any invertible $\mathcal{O}$-module L consider a morphisms of $\mathcal{O}$-modules $i_T : L \to L(T)$. This is an imbedding of $\mathcal{O}$-modules and we will denote by $\mathcal{P}(L, T)$ the quotient $\mathcal{O}$-module.

4. Check that the $\mathcal{O}$-module $\mathcal{P}(L, T)$ is a sum of skyscraper sheaves at points $a \in T$, such that the stalk of this sheaf at the point $a \in T$ is **canonically** isomorphic to a one dimensional k-vector space $\hat{L}(a) := T_a(C) \otimes L|_a$. In particular the space of global sections $P(L, T) = \Gamma(C, \mathcal{P}(L, T))$ is isomorphic to $\bigoplus_{a \in T} \hat{L}(a)$.

5. Consider exact sequence of morphisms of $\mathcal{O}$-modules

(*) $0 \to L \to L(T) \to \mathcal{P}(L, T) \to 0$

Let us apply to it the functor Γ and get a sequence of morphisms of vector spaces

(**) $0 \to \Gamma(L) \to \Gamma(L(T)) \to P(L, T)$

(i) Show that this sequence is exact

(ii) Denote by $H_T(L)$ the cokernel of the morphism $\nu_T : \Gamma(L(T)) \to P(L, T)$.

(iii) Show that $\dim H_T(L) = h(L) - h(L(T))$. In particular, if $h(L(T)) = 0$ (for example this happens if $deg(T) \gg 0$) then $\dim H_T(L) = h(L)$.

6. If $T \leq T'$ we have canonical imbedding $P(L,T) \to P(L,T')$.
(i) Show that it induces a morphism $H_T(L) \to H_{T'}(L)$.
(ii) Show that this morphism is always mono.

Using Problem 6 we define $H(L) := \lim_T H_T(L)$.
For every T we have an imbedding $H_T(L) \to H(L)$. Problem 5 implies that this is an isomorphism if $h(L(T)) = 0$. In particular this is an isomorphism when $deg(T)$ is very large.

7. Fix a point $a \in C$. Suppose there exists a function $f \in k(C)$ that is regular outside of a and has pole of order exactly 1 at the point a.
Show that f defines an isomorphism of the curve C with $\mathbf{P}^1$. In particular in this case C has genus 0.

For any integer d we denote by $Pic^d(C) \subset Pic(C)$ the set of divisors of degree d (modulo principle divisors).

8. Let C be a curve of genus 1.
(i) Show that in this case the canonical map $C \to Pic^1(C)$ is a bijection.
(ii) Let us fix a point $a \in C$. Show that the set of points of C has a canonical structure of an abelian group such that the point a is its zero element.

9. Let C be a curve of genus g. For any $d \geq 0$ consider an algebraic variety $Sym^d(C)$ obtained as a quotient of the variety $C^d = C \times C \times ... \times C$ by action of the symmetric group Sym_d interchanging the factors.
We have a natural map $\nu_d : Sym^d(C) \to Pic^d(C)$ given by $\nu_d(x_1, ..., x_d) = \mathcal{O}(x_1 + x_2 + ... + x_d)$.
(i) Show that if $d \geq g$ the map ν_d is surjective. Show that its fibers are isomorphic to projective spaces.

(ii) Show that when $d > 2g - 2$ all fibers have the same dimension.

(iii) Show that when $d = g$ the map ν_d is almost bijective. Namely there exists an open dense subset $U \subset Sym^d(C)$ on which the map ν_g is injective.

(*) (iv) Use this to define a structure of a connected algebraic group on the group $Pic^0(C)$ such that ν_g will be a morphism of algebraic varieties. Show that the morphism ν_d is submersive for $d > 2g - 2$.

10. Let us fix a line bundle (i.e. an invertible $\mathcal{O}$-module) R on C of maximal degree such that $h(R) \neq 0$. Show that in this case $h(R) = 1$ and hence the space $H(R)$ is one dimensional.
Let us fix an isomorphism $i : H(R) \simeq k$.
(i) Show that for every point $a \in C$ the fiber R_a is **canonically** isomorphic to $T^*C = \Omega|_a$.
Later we will see that this defines a canonical isomorphism $R \simeq \Omega$, that depends on the choice of the isomorphism i. It is easy to see that this gives a canonical isomorphism $H(\Omega) = k$, independent on any choices.

(ii) For any line bundle $\mathcal{L}$ consider the line bundle $\hat{\mathcal{L}} := \mathcal{L}^* \otimes \Omega$.
Define a canonical pairing B between $L(\mathcal{L})$ and $H(\hat{\mathcal{L}})$. Show that the pairing B is well defined and nondegenerate on $L(\mathcal{L})$. Show that B is nondegenerate.