Problem assignment 12.

Algebraic Geometry and Commutative Algebra

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1. Let X be a topological space and $0 \to L \to M \to N \to 0$ a s.e.s (short exact sequence) of sheaves on X. Let us fix a section $\xi \in N(X)$; we would like to find its lifting to M, i.e. a section $\delta \in M(X)$ such that $p(\delta) = \xi$ (here p denotes the morphism $p : M \to N$ above).

Suppose X is a union of two open subsets U and W. Suppose we found sections δ_U and δ_W over these subsets.

(i) Show that if L is flabby then there exists a section $\delta \in M(X)$ such that $p(\delta) = \xi$ and $\delta|_U = \delta_U$

(ii) Show that in general if the sheaf L is flabby then the morphism $p : M(X) \to N(X)$ is onto.

Hint. Use Zorn's lemma.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an additive functor (you may always assume that these categories are some categories of modules).

2. (i) Show that the functor F is exact iff it maps s.e.s into s.e.s.

(ii) We say that the functor F is **left exact** if it maps any left short exact sequence $0 \to L \to M \to N$ into a left exact sequence.

Show that F is left exact iff it maps any s.e.s into left exact sequence.

Some cohomological constructions.

Consider the category of complexes $Com(\mathcal{A})$. Usually we denote complex as $C^\cdot$ meaning $... \to C^i \to C^{i+1} \to ....$

For every complex $C^\cdot$ we denote by $H^i(C^\cdot)$ its cohomology groups. We say the $C^\cdot$ is acyclic (= exact) at place i if $H^i(C^\cdot) = 0$. Similarly for any collection of places i.

Check that $Com(\mathcal{A})$ is an abelian category. We usually identify an object $F \in \mathcal{A}$ with the complex $F^\cdot$ that has F in place 0 and 0 at all other places.

3. (i) Show that a morphism of complexes $\nu : C^\cdot \to D^\cdot$ induces morphisms of cohomology groups $\nu_* : H^i(C) \to H^i(D)$.

(ii) Let $0 \to A^\cdot \to B^\cdot$
$cdot \to C^\cdot \to 0$ be a short exact sequence of complexes.

Construct connecting morphisms $\delta^i : H^i(C) \to H^{i+1}(A)$ and show that the long sequence of cohomologies is exact.

Show that the construction of the connecting morphisms is functorial.

Definition. A morphism of complexes $\nu : C^\cdot \to D^\cdot$ is called **quasiisomorphism** if it induces an isomorphism on cohomologies.

4. Prove **Five lemma.** Let $L^\cdot, M^\cdot$ be two complexes and $\nu : L^\cdot \to M^\cdot$ be a morphism of complexes.

Let us assume that the complexes are exact, morphisms ν_1 and ν_{-1} are isomorphisms, ν_{-2} is epimorphic and ν_2 is mono.

Sow that in this case the morphism ν_0 is an isomorphism.

Definition. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. A collection of objects $Q \subset ISO(\mathcal{A})$ we call **adapted to** F if it satisfies the following conditions

(ad1) for any s.e.s $0 \to L \to M \to N \to 0$ we have

(i) If $L, M \in Q$ then $N \in Q$

(ii) If $L \in Q$ then $0 \to F(L) \to F(M) \to F(N) \to 0$ is a s.e.s.

(ad2) The family Q is rich, i.e. every object can be imbedded into an object $D \in Q$.

[P] 5. Show that using the family Q adapted to F we can construct the right derived cohomological functor $RF = \{R^i F\}$.

Hint. Show that any object M has a right Q-resolution R. Show that for any two Q resolutions R, R' of M there exists a Q resolution R'' that contains R and R'. Show that in this case the imbedding $R \to R''$ induces a quasiisomorphism $F(R) \to F(R'')$ (and similarly for R').

Show that for any morphism $\nu : M \to N$ and any Q resolution R of M we can find a Q resolution P of N and extend ν to a morphism of resolutions $\nu' : R \to P$.

Definition. **Cone and cocone constructions.** Let $\nu : L \to M$ be a morphism of complexes.

(i) We construct a new complex $C(\nu)$ – it is called the **cone** of the morphism ν – as follows. We extend ν to a complex of complexes placing $L^{\cdot}$ and $M^{\cdot}$ in places -1 and 0, consider the corresponding bicomplex B and set $Cone(\nu) := Tot(B)$.

We define **cocone** $CC(\nu)$ of the morphism ν by $CC(\nu) = C(\nu)[-1]$.

[P] 6. (i) Write explicit formulas for the complexes $C(\nu)$ and $CC(\nu)$. Show that there exist short exact sequences of complexes $0 \to M \to C(\nu) \to L[1] \to 0$ and $0 \to M[-1] \to CC(\nu) \to L \to 0$

Deduce from this a long exact sequence connecting cohomologies of L, M and $C(\nu)$.

(ii) Show that the morphism of complexes ν is a quasiisomorphism if an only if the complex $C(\nu)$ is acyclic.

(ii) Show that if ν is injective then $C(\nu)$ is quasiisomorphic to the quotient complex M/L, and if ν is surjective then $CC(\nu)$ is quasiisomorphic to the complex $K = Ker(\nu)$.

Let X be a topological space and $j : U \to X$ an imbedding of an open subset. We define a functor $R_U : Com(SH(X)) \to Com(Sh(X))$ as follows:

For every sheaf F on X we consider the sheaf $F_U := j_*(F|_U)$ and the natural morphism $\nu_U : F \to F_U$. Then we ex tend this construction to complexes of sheaves and set $R_U(F^{\cdot}) := CC(\nu_U)$. We denote by p the natural epimorphism of complexes morphism $R_U(F^{\cdot}) \to F^{\cdot}$.

7. (i) Show that the complex $R_U(F^{\cdot})$ is acyclic on U.

(ii) Show that if $U, V \subset X$ are open subsets then $R_U R_V(F^{\cdot}) \approx R_V R_U(\mathbf{F}^c dot)$.

8. Given a finite collection of open subsets $\mathcal{U} = (U_1, ..., U_l)$ we define for any sheaf F a complex of sheaves $R_{\mathcal{U}}(F) := R_{U_1} R_{U_2} ... R_{U_l}(F)$ and a natural surjective morphism $p : R_{\mathcal{U}}(F) \to F$ (here we consider F as a complex of sheaves concentrated in degree 0).

(i) Write down this explicitly. Show that p is an epimorphism of complexes. Show that $R_{\mathcal{U}}$ is acyclic on the union of sets U_i.

Definition. Suppose $\mathcal{U}$ is a covering. Define a **Čech resolution** $\mathcal{C}(F)$ of the sheaf F by formula $Cech_{\mathcal{U}}(F) := Ker p : R_{\mathcal{U}}(F) \to F[1]$. Define the **Čech complex** of the sheaf F as a complex of abelian groups $C^{\cdot}(F) := \Gamma(\mathcal{C}^{\cdot}(F)$.

8. Write down explicit formulas for the Čech resolution and the Čech complex of F. Show that the Čech complex is a resolution of the sheaf F.

9. Let X be a quasicompact topological space, $\mathcal{B}$ a basis of topology on X closed under finite intersection. Let F be a sheaf on X that satisfies the following acyclicity condition:

(*) Let $B \in \mathcal{B}$ be a basic open subset. Then the restriction $H = F|_B$ is Čech acyclic. This means that for every finite open covering $\mathcal{U}$ of B by basic open subsets $B_i \in \mathcal{B}$ the complex $\Gamma(R_{\mathcal{U}}(H))$ is acyclic.

Show that then the restriction H of the sheaf F to any basic subset $B \in \mathcal{B}$ is Γ acyclic.

Hint. Consider s.e.s of sheaves on $\mathcal{P} = 0 \to F \to I \to G \to 0$ on X where I is flabby. Show that for every basic B the corresponding sequence of sections $\Gamma(B, \mathcal{P})$ is exact. Deduce from this that the sheaf G is Čech acyclic. Then use induction.