Problem assignment 16.

Algebraic Geometry and Commutative Algebra 2.

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1. Fix an abelian group A.

(i) Show that a function f on A is polynomial of degree $\leq d$ iff for any lattice (i.e. group isomorphic to $\mathbf{Z}^n$ for some n) L and any morphism $\nu : L \to A$ the function $\nu^*(f)$ on L is polynomial in the usual sense.

(ii) Fix $d \geq 0$. Given a function $f \in Pol^{\leq d}(A)$ we construct a function $R = R_f$ in d variables $a_i \in A$ via $R(a_1, a_2, ..., a_d) = \Delta_{a_1} \circ \Delta_{a_2} \circ ... \circ \Delta_{a_d}(f)$ (this makes sense since this function is constant).

Show that the function R_f is a multilinear symmetric function.

(iii) Let us consider the function $Q(x)$ on A defined by $Q(x) = R(x, x, ..., x)$ (the diagonal part of R). Show that the function $h(x) := d!f(x) - Q(x)$ lies in $F^{d-1}(A)$.

Remarks. (a) The function Q is a homogeneous polynomial functions on A and the form R is its polarization.

(b) The problem 1(iii) provides a useful computational tool for computing the function Q in terms of f. Namely $Q(x) = d!\ \lim_{k\to\infty}(f(kx)/k^d)$.

II. Intersection theory on a smooth surface.

We will be interested in the following situation. We fix a smooth connected projective surface S (i.e. variety of dimension 2) and set $A = Pic(S)$. We consider the function χ on A given by $\chi(L) =$ Euler characteristic of L.

This is a quadratic function. We denote by B the corresponding symmetric bilinear form and by Q the corresponding quadratic form.

We denote by $Div(S)$ the abelian group of divisors of S. To every divisor $E \subset S$ we associate an invertible $\mathcal{O}_S$ module $L = \mathcal{O}(E) \in Pic(S)$. This defines a projection $p : Div(S) \to Pic(S)$ that identifies the group $Pic(S)$ with the quotient group $Div(S)/PrinDiv$. We would like to show that the intersection theory of smooth divisors can be carefully described in term the form B on $Pic(S)$ described above.

2. (i) Let E be a smooth divisor on S and L invertible $\mathcal{O}$-module. Show that $B(L, \mathcal{O}(E)) = \deg L|_E$.

(ii) Let E, F be two smooth divisors on S. Suppose they intersect transversally. Show that the number of intersection points $< E, F >$ equals to $B(\mathcal{O}(E), \mathcal{O}(F))$.

Remark. In fact this is true even if the divisors E and F are not smooth provided that they are smooth and transversal at all intersection points.

More generally, if E, F are two divisors that have finite intersection we can define the local multiplicities of intersection points like in Bezout theorem; in this case the sum of these multiplicities equals $B(\mathcal{O}(E), \mathcal{O}(F))$.

3. Let $E \subset S$ be a smooth divisor, $L = \mathcal{O}(E) \in Pic(S)$. Show that $Q(L) = \deg N_C(S)$, where $N_C(S)$ is the normal bundle of C in S.

Hint. Solution 1. Show that the invertible $\mathcal{O}$-module $\mathcal{O}(E)|_E$ is isomorphic to $N_C(S)$.

Hint. Solution 2. Show that $\mathcal{O}_S$-module $L^{\otimes k}$ is glued from $\mathcal{O}$-module $\mathcal{O}$ and $\mathcal{O}_S$ modules $i_*(N^j)$ for $j = 1, ..., k$, where $i : E \to S$ is the imbedding and N^j are tensor powers of the normal bundle $N = N_E$ on the curve E. Estimate Euler characteristics of all these sheaves up to constant in terms of degrees of these tensor powers.

Definition. Let $B(x, y)$ be a symmetric bilinear form on the group A with values in $\mathbf{Z}$. We denote by $Q = Q_B$ the corresponding quadratic function $Q(x) := B(x, x)$.

We say that the form B is positive if for all $x \in A$ we have $B(x, x) \geq 0$; we say that B is negative if the form $-B$ is positive.

One of the basic highly non-trivial results that underscores the intersection theory on S is the following

Hodge Index Theorem Fix a very ample bundle $H \in Pic(S)$. Then $B(H,H) > 0$ and the form B is **negative** on the orthogonal complement to H in $Pic(S)$

We will prove this statement later. Here we will prove it in the following special case.

Theory of correspondences.

Let us fix two smooth projective curves C and D and consider the surface $S = C \times D$. Set $A = Pic(S)$ and consider the function χ on the group $A = Pic(S)$, bilinear form B and quadratic form Q on the group A defined above.

Fix a point $c \in C$ and denote by h an element in A corresponding to the divisor $c \times D$ (horizontal fiber). Similarly fix a point $d \in D$ and denote by v the vertical fiber.

We would outline the proof of the following

Theorem. Let A' denote the orthogonal complement of v is A. Then the restriction of the bilinear form B to the subgroup A' is negative.

In order to prove this claim it is enough to check the following

Proposition. Consider the function $f = \Delta_v(\chi)$ on the group A and denote by H the set of zeroes of f (this is a " hyperplane" parallel to A'). Then for any $L \in H$ we have $\chi(L) \leq 0$.

4. (i) Check that $B(v,v) = B(h,h) = 0$ and $B(h,v) = 1$.

(ii) Check that for any $L \in Pic(S)$ we have $f(L) = \chi(L|v)$ (restriction of L to the vertical fiber).

(iii) Check that H is parallel to A' and show that the proposition implies the Theorem.

Step 1. Show that we can assume that for the restriction L_v of the module L to the vertical fiber satisfies

$l(L_v) = h(l_v) = 0$

Hint. Show that there exists an invertible $\mathcal{O}$-module M on C such that $l(M) = h(M) = 0$. Then

consider $\mathcal{O}_C$- module $M' = M \otimes (L|v)^{-1}$ and show that we can replace L by $L' = L \otimes pr^*(M')$ since $\chi(L) = \chi(L')$.

The equality of Euler characteristics follows from the fact that the $\mathcal{O}$-module M' has degree 0 and hence can be included in a connected flat family of $\mathcal{O}$-modules on C that contains the trivial module $\mathcal{O}_C$.

Step 2. Consider the projection $p : S \to D$.

(i) Check that $R^i p_*(L) = 0$ for $i > 1$ and that $R^1 p_*(L)$ has fiber 0 at the point d.

In particular $R^1 p_*(L)$ is a torsion sheaf equal to 0 in some neighborhood U of d.

(ii) Check that $p_*(L) = R^0 p_*(L)$ has fiber 0 at d. Deduce from this that it is equal to 0 on C.

Step 3. Using the fact that $R^0 p_*(L) = 0$ and $R^1 p_*(L)$ is a torsion sheaf and hence has only cohomologies in degree 0 deduce that $H^i(S, L) = 0$ when $i \neq 1$. This shows that $\chi(L) \leq 0$, QED.

5. Show that if $L \in H$ and $\chi(L) = 0$ then in fact L lies in $Pic(C) \cdot Pic(D) \subset Pic(S)$.