Problem assignment 1.

Algebraic Theory of D-modules.

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A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Dmitry).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign $(\Box)$marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

1. (i) Prove that the algebra D_n of differential operators on R^n with polynomial coefficients is generated by $x_1, ..., x_n, \partial_1, ..., \partial_n$ with relations $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, $[\partial_i, x_j] = \delta_{ij}$.

(ii) Prove that the monomials $x^\alpha \partial^\beta$ form a basis of D_n.

Some problems about Generalized functions.

2. Here is the definition of topology on $C_c^\infty(\mathbf{R}^n)$ in terms of sequences:
$\phi_i \to \phi$ iff
(i) There exists a compact subset $K \subset X$ such that for any i, $supp(\phi_i) \subset K$

(ii) For any differential operator d with constant coefficients the sequence of functions $d\phi_i$ converges to the function $d\phi$ absolutely and uniformly.

The task is to define this topology in terms of open sets.

3. Let $Dens_c^\infty(\mathbf{R}^n)$ denote the space of smooth densities with compact support.

Define the natural pairing between spaces $C_c^\infty(\mathbf{R}^n)$ and $Dens_c^\infty(\mathbf{R}^n)$.

Show that the completion of the space $C_c^\infty(\mathbf{R}^n)$ in weak topology defined by this pairing is isomorphic to the space of generalized functions $GF(\mathbf{R}^n)$.

4. i) Define multiplication of a generalized function by a smooth function.

ii) Define partial derivatives of generalized functions. Show that for $\mathbf{C}^1$-functions it agrees with the natural definition.

iii) For any differential operator d with smooth coefficients describe explicitly the action of d on generalized functions in terms of distributions.

5. Let P be a real valued polynomial functions on $X = \mathbf{R}^n$. Fix a connected component Θ of the open set $X^* = X \setminus Zeroes(P)$ and consider a function P_Θ^λ that is 0 outside Θ and equals to $|P(x)|^\lambda$ inside Θ.

Show that when $Re\lambda > n$ this is a locally integrable function. We denote by the same symbol the corresponding generalized function.

Show that as generalized function P_Θ^λ expends to a meromorphic function with values in the space $GF(\mathbf{R}^n)$ to the whole complex plane.

6. Let P, Q be real valued polynomial functions on X. Prove meromorphic continuation of the family of generalized functions $P^\lambda Q^\mu$.

7. Prove **Kaplansky Lemma**.

Lemma. Let k be an uncountable algebraically closed field, V a countable dimensional vector space over k and $T : V \to V$ a linear operator. Then its specter is not empty, i.e. there exists a $\lambda \in k$ such that the operator $T - \lambda$ is on invertible.

Hint. Show that otherwise the space V is a vector space over the field of rational functions $k(t)$ that has uncountable dimension over the field k.