Problem assignment 3.

Algebraic Theory of D-modules.

Joseph Bernstein November 28, 2011.

Definition. Let X be a smooth algebraic variety, F a D_X-module. We say that F is **smooth** (equivalent term $\mathcal{O}$-coherent) if it is coherent as $\mathcal{O}_X$-module.

1. (i) Suppose $\dim X = 1$. Show that every smooth D_X-module F is locally free as $\mathcal{O}$-module. (**Hint.** Describe the torsion submodule of F).

(ii) Prove that in any dimension smooth D_X-module F is locally free as $\mathcal{O}$-module (**Hint.** Look at dimensions of fibers of F).

(iii) Show that full subcategory of smooth D_X-modules is naturally equivalent to the category of vector bundles (of finite rank) on X equipped with flat connection ∇.

2. Let F be a holonomic D_X-module. Show that there exists a dense open subset $U \subset X$ such that the restriction of F to U is a smooth D-module.

3. (i) Let F be a coherent D_X-module. Show that there exists an $\mathcal{O}$-coherent submodule $P \subset F$ that generates F as a D-module.

(ii) Show that every coherent D_X-module F is Noetherian, i.e. every D-submodule of F is coherent.

(iii) Let F be a D_X-module and U an open subset of X. Let G be a coherent D-submodule of the restricted D_U-module $F|_U$. Show that it can be extended to a coherent D_X-submodule $H \subset F$ (i.e. the restriction $H|_U$ coincides with G).

(iv) Show that any D_X module F is a union of coherent D_X submodules.

4. (i) Let $C^\cdot$ be a bounded complex of D_X-modules. Suppose that all its cohomologies are coherent. Show that there exists a bounded complex $G^\cdot$ of coherent D_X-modules and a quasiisomorphism $i : G^\cdot \to F^\cdot$

(ii) Show that the natural functor $D^b(Coh(D_X) \to D^b_{coh}(\mathcal{M}(D_X))$ is an equivalence of triangulated categories.

Definition. We would like to define the truncation functor $\tau_{\leq 0} : Com(A) \to Com(A)$ and a morphism of functors $i : \tau_{\leq 0} \to Id$.

Namely, for every complex X consider the subcomplex $\tau_{\leq 0}(X) \subset X$ such that

it is 0 in degrees > 0, it coincides with X in degrees < 0 and it is equal to the kernel of the differential d in degree 0

5. (i) Show that i gives an isomorphism of cohomology groups H^k for $k \leq 0$.

(ii) Show that the complex $\tau_{\geq 1}(X) := X/\tau \leq 0(X)$ is acyclic in degrees ≤ 0 and has the same cohomologies as X in degrees > 0.

Functors $\tau_{\leq 0}$ and $\tau_{>0}$ are called **truncation functors**. Similarly we define truncation functors $\tau_{\leq d}$ and $\tau_{>d}$ for any integer d.

(iii) Show that truncation functors define functors in derived category.

(iv) Compute the quotient complex $\tau_{\leq 0}/\tau_{\leq =1}$ as an object in derived category.

6. Show that the category A is a naturally equivalent to the full subcategory $D^0(A)$ of the derived category $D(A)$ consisting of objects X such that $H^k(X) = 0$ for $k \neq 0$

7. Let B be a bounded bicomplex. Suppose we know that the differential d_2 is acyclic outside the 0 row-th. Denote by H the complex consisting of objects $H_i := H^0(Col_i(B)$ with differential d_1.

Show that the complex H is quasiisomorphic to the complex $Tot(B)$.

Hint. First consider the case when $H = 0$ (Grothendieck lemma). Then consider the case when $B_{ij} = 0$ for $j > 0$ and reduce it to the previous case by applying vertical truncation functor to the bicomplex B. Then reduce the general case to this one using another vertical truncation.