Problem assignment 4.

Algebraic Theory of D-modules.

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In this assignment we assume that an algebraic variety X is quasiprojective. In particular this implies that any coherent $\mathcal{O}_X$-module is a quotient of a coherent locally free $\mathcal{O}_X$-module.

1. Let X be an algebraic variety, $Z \subset X$ its closed subset and $U = X \setminus Z$ its open complement. We denote by j the open imbedding $j : U \to X$.

(i) Describe the functors $j_0 : \mathcal{M}(\mathcal{O}_U) \to \mathcal{M}(\mathcal{O}_X)$ and $\Gamma_Z : \mathcal{M}(\mathcal{O}_X) \to \mathcal{M}_Z(\mathcal{O}_X)$.

Show that these functors map injective objects into injective. Prove that categories $\mathcal{M}(\mathcal{O}_X)$ and $\mathcal{M}_Z(\mathcal{O}_X)$ have enough injective objects.

(ii) Establish for any $R \in D(\mathcal{O}_X)$ the exact triangle $R_Z \to R \to R_U$.

(iii) Explain how to split the category $D(\mathcal{O}_X)$ as an extension of categories $D_Z(\mathcal{O}_X)$ and $D(\mathcal{O}_U)$ (do this first for the category D^+, then think about the general case).

2.(i) Consider the functor of sections $\Gamma : \mathcal{M}(\mathcal{O}_X) \to \mathrm{Vect}$. Show that the cohomological dimension of this functor is $\leq \dim X$. In other words for any $\mathcal{O}_X$-module R the derived functors $H^i(X, R) = R^i\Gamma(R)$ vanishes if $i > \dim X$.

(ii) Show that in fact $H^i(X, R) = 0$ if $i > \dim \mathrm{supp}(R)$.

(iii) Fix a coherent locally free $\mathcal{O}_X$-module F. Show that the cohomological dimension of the functor $Hom_F : \mathcal{M}(\mathcal{O}_X) \to Vect$ given by $R \mapsto Hom(F, R)$ is bounded by $\dim X$.

Show that in this case for any R we have $Ext^i(F, R) = 0$ for $i > \dim \mathrm{supp}(R)$.

3. Fix a coherent $\mathcal{O}$-module F on X.

(i) Show that for any $\mathcal{O}_X$-module R we can define a $\mathcal{O}_X$-module $\mathcal{H}om(F, R)$ by condition that on every open affine subset U we have $\mathcal{H}om(F, R)(U) = Hom_{\mathcal{O}(U)}(F(U), R(U))$.

This defines a functor (inner Hom) $\mathcal{H}om_F : \mathcal{M}(\mathcal{O}_X) \to \mathcal{M}(\mathcal{O}_X)$ by $R \mapsto \mathcal{H}om(F, R)$.

(ii) Show that $Hom(F, R) = \Gamma(\mathcal{H}om(F, R))$

In problems bellow we assume that X is a smooth quasiprojective variety of dimension n.

4. (i) Show that the cohomological dimension of the category $\mathcal{M}(\mathcal{O}_X)$ is bounded by $2n$.

Hint. You have to show that $Ext^i(F, R) = 0$ for $i > 2n$. Reduce this to the case when F is coherent. Show that F has a resolution by locally free modules of length $\leq n$. Then use previous problems.

(ii) Let us fix a coherent $\mathcal{O}_X$-module F. Consider the derived functor $\mathcal{E}xt_F : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$, $R \mapsto \mathcal{E}xt(F, R)$ (we consider everywhere bounded derived categories). Show that the derived functor Ext_F defined by $R \mapsto Ext(F, R)$ is isomorphic to the composition of functors $R\Gamma \circ \mathcal{E}xt_F(R)$.

(*) **5.** Show that the cohomological dimension of the category $\mathcal{M}(\mathcal{O}_X))$ is bounded by n.

Hint. Show that it is enough to consider spaces $Ext^i(F, R)$ for the case when F is coherent and R is coherent and locally free.

Show that in this case $\mathcal{E}xt(F, R)$ is glued from shifts of the $\mathcal{O}$-modules $\mathcal{E}xt^i(F, R)$, and every such $\mathcal{O}$-module has dimension of support $\leq n - i$

6. (i) Show that the category $\mathcal{M}(\mathcal{D}_X)$ has enough injective objects. Show that it has enough locally projective objects.

(ii) Explain how to compute $Ext^i(F^\cdot, H^\cdot)$ using resolutions in semi-functorial way (i.e. functorial with respect to a finite number of objects and morphisms).

(iii) Using this show that the cohomological dimension of the category $\mathcal{M}(\mathcal{D}_X)$ is bounded by $3n$ (later we will see that it is bounded by $2n$).

7. Let T be a singular quasiprojective algebraic variety.

(i) Show that the category $\mathcal{M}(\mathcal{D}_T)$ has enough injective objects. Show that this category has finite cohomological dimension.

(ii) Fix a closed imbedding of T into a smooth variety X. Show that the category $D(\mathcal{D}_T) := D^b(\mathcal{M}(\mathcal{D}_T))$ is naturally equivalent to the category $D^b_T(\mathcal{M}(\mathcal{D}_X))$ - derived category of bounded complexes of $\mathcal{D}_X$-modules acyclic outside T.

8. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F : D(\mathcal{A}) \to D(\mathcal{B})$ a strongly exact functor. We call an object $R \in \mathcal{A}$ F**-acyclic** if $F(R) \in \mathcal{B} \subset D(\mathcal{B})$.

(i) Let $M^{\cdot}$ be a bounded complex in $Com(\mathcal{A})$ consisting of F-acyclic objects and let M' denote its image in the derived category $D(\mathcal{A})$.
Show that the complex $F(M^{\cdot}) \in Com(\mathcal{B})$ represents the object $F(M') \in D(\mathcal{B})$.

(ii) We would like to compare two strongly exact functors $F, G : D(\mathcal{A}) \to D(\mathcal{B})$. Suppose we found some full subcategory $\mathcal{Q} \subset \mathcal{A}$ such that all its objects are F and G acyclic.
Show that if the category $\mathcal{Q}$ is rich enough then $Hom(F, G) = Hom(F|_{\mathcal{Q}}, G|_{\mathcal{Q}})$.

(iii) Show that under some conditions on the subcategory $\mathcal{Q}$ we can extend a functor $F : \mathcal{Q} \to \mathcal{B}$ to a strongly exact functor $F : D(\mathcal{A}) \to D(\mathcal{B})$.

9. Let $\mathcal{A}$ be an abelian category. Consider a bifunctor $Hom : \mathcal{A}^o \times \mathcal{A} \to Ab$ given by $(M, N) \mapsto Hom_{\mathcal{A}}(M, N)$.

(i) Suppose the category $\mathcal{A}$ has enough injective objects. Show how for fixed M to define derived functors of the functor Hom; these functors are denoted $Ext^i(M, N)$.

(ii) Similarly define derived functors of Hom with respect to M (for example you can assume that $\mathcal{A}$ contains many projective objects).
Show that these derived functors are isomorphic to $Ext^i(M, N)$.
In other words one can derive functors in each variable and the result if the same.

(iii) Show that for any abelian category $\mathcal{A}$ the functors $Ext^i(M, N)$ are well defined - one does not need to have many injective or projective objects.
Hint. Show that $Ext^i(M, N) = Hom_{D(\mathcal{A})}(M, N[i])$.

10. Using homological methods prove the **Principle Ideal Theorem**.

Theorem. Let X be an irreducible algebraic variety of dimension n, f a non-zero regular function on X. Let Z denote the closed subvariety defined by equation $f = 0$.
Show that every irreducible component of Z has dimension $n - 1$.