Problem assignment 1.

Algebra B3 – Commutative Algebra

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A remark on problems in different areas. In my assignments I will try to single out problems that are not really about commutative algebra. For example sign (LA) signifies a problem (or a definition) from linear algebra, (Top) for topology.

A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them, but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Erez).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always directly needed in the course, but I definitely advise you to think about these problems.

Without further specification by algebra we mean a commutative algebra with 1.

1. (i) Show that in any algebra we have identities $0x \equiv 0$, $a(-x) \equiv -ax$.

(ii) Give an example of algebras A, B and a map $\nu : A \to B$ that preserves addition and multiplication, but does not map 1_A into 1_B.

Such maps are **not** called the homomorphism of algebras.

2. Let $\nu : A \to B$ be a morphism of algebras. Show that the kernel $J = Ker(\nu)$ is an ideal of A, the image $S = Im(\nu)$ is a subalgebra of B and that the algebra S is naturally isomorphic to A/J.

3. (i) Let A be an algebra. Show that the set $Nil(A)$ of nilpotent elements of A is an ideal of A. Show that the quotient algebra $B = A/Nil(A)$ does not have nilpotents.

[**P**] (ii) Show that the set $1 + Nil(A)$ is a subgroup of the multiplicative group A^*. (As usual A^* denotes the group of units, i.e. invertible elements, of the algebra A). Show that the quotient group $A^*/(1+Nil(A))$ is naturally isomorphic to the multiplicative group B^*.

4. Let A be an algebra. Show that the following conditions are equivalent

(i) A is a domain, i.e. it does not have zero divisors.

(ii) There exists an imbedding $\nu : A \to K$ where K is a field

(iii) There exists an imbedding $\nu : A \to K$ where K is an algebraically closed field.

Localization. Let A be an algebra. Fix an element $f \in A$. We say that a morphism of algebras $\nu : A \to B$ is **f-inverting** if the element $\nu(f) \in B$ is a unit.

Definition. Let $i : A \to D$ be an f-inverting morphism. We say that it has **universal property with respect to f-inverting morphisms** if for any f-inverting morphism $\nu : A \to B$ there exists unique morphism $\lambda : D \to B$ that makes the diagram commutative (i.e. $\lambda \circ i = \nu$).

Proposition. (i) Let i, i' be two f-inverting morphisms that satisfy the universal property. Then they are **canonically** isomorphic.

5. Theorem. For any algebra A and an element $f \in A$ there exists an f-inverting morphism $i : A \to A_f$ satisfying the universal property.

From the proposition we see that the algebra A_f and the morphism $i : A \to A_f$ are uniquely defined **up to canonical isomorphism**. The algebra A_f is called the localization of A with respect to the element f.

Hint. Consider the set of fractions $F = \{\frac{a}{f^k}\}$, where $a \in A, k \in \mathbf{Z}_{\geq 0}$.

Define the equivalence relation $\sim$ on this set by $\frac{a}{f^k} \sim \frac{b}{f^l}$ if there exists m such that $f^m(f^l a - f^k b) = 0$ in A.

Show that this is an equivalence relation. Define A_f to be the quotient set $A_f := F/\sim$. Define the algebra structure on the set A_f and show that the natural morphism $i : A \to A_f$ is f-inverting and has the universal property.

[P] 6. (i) Show that $A_f \simeq A[t]/(1 - ft)$.

(ii) Describe the kernel of the canonical morphism $i : A \to A_f$. Show that the algebra A_f is trivial iff f is nilpotent.

We have proven in class the following

Theorem. Let A be a non-trivial algebra. Then it has a maximal ideal $\mathfrak{m} \subset A$.

[P] 7. Exercises on Zorn's lemma.

(i) Let V be a vector space over a field K. Show that it has a basis.

(ii) Let R be an algebra (maybe not commutative). Let M be a non-zero finitely generated (left) R-module. Show that it has an R-submodule L such that the quotient module $N = M/L$ is a (non-zero) simple R-module.

Show that the analogous statement is not true if we do not assume that M is finitely generated.

(iii) Let M be an R-module. Fix a subset $Q \subset M$ that does not contain 0. Consider all R-submodules in M that do not intersect Q. Show that there exists a maximal submodule with this property.

[P] 8. Consider an element $f \in A$.

(i) Show that the element f is nilpotent iff for any morphism $\nu : A \to K$ into a field we have $\nu(f) = 0$.

Show that the element f is a unit iff for any morphism $\nu : A \to K$ into a field we have $\nu(f) \neq 0$.

(ii) Show that the intersection of all prime ideals of A equals to the ideal $Nil(A)$. Can you describe the union of all prime ideals ?

(iii) For any ideal $J \subset A$ defines its radical $Rad(J) \subset A$ and show that it is an ideal. Show that $Rad(J)$ is the intersection of all prime ideals containing J.

Let $\nu : A \to B$ be a morphism of algebras. We define contraction map of sets $\nu^* : Ideals(B) \to Ideals(A)$ by $\nu^*(J) := \nu^{-1}(J)$. We define extension map of sets $\nu_* : Ideals(A) \to Ideals(B)$ by $\nu_*(I) := B\nu(I)$ (describe precisely this ideal).

[P] 9. (i) Show that contraction defines a map $\nu^* : Spec(B) \to Spec(A)$ (Here $Spec(A)$ is the set of prime ideals of A).

(ii) Consider the localization morphism $i : A \to A_f$. Show that in this case the composition $i_* \circ i^*$ is identity. Hence the map i^* identifies the set $Ideals(A_f)$ with a subset of the set $Ideals(A)$. Describe this subset explicitly.

(iii) Show that the subset $Spec(A_f) \subset Spec(A)$ coincides with the set of prime ideals that do not contain f.

[P] 10. Let $B = A[t]$ be the algebra of polynomials and $\nu : A \to B$ the natural imbedding.

(i) Show that if A is a domain then B is also a domain.

Show that the extension morphism $\nu_* : Ideals(A) \to Ideals(B)$ maps prime ideals into prime ideals.

(ii)Consider a polynomial $f \in B$, $f = \sum a_i t^i$.

Show that f is nilpotent iff all coefficients a_i are nilpotent.

Show that f is a unit iff the coefficient a_0 is a unit and all other coefficients are nilpotent.

Hint. Prove claim (ii) for the case when A is a field and then use problem 8.