Problem assignment 2.

Algebra B3 – Commutative Algebra

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A remark on different kinds of problems. In all my home assignments I will use the following system.

The problems without marking are just exercises. You have to convince yourself that you can do them, but it is not necessary to write them down (if you have difficulties with one of these problems ask me or Erez).

The problems marked by [**P**] you should hand in for grading.

The sign $(*)$ marks more difficult problems.

The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always directly needed in the course, but I definitely advise you to think about these problems.

Without further specification by algebra we mean a commutative algebra with 1**.**

Hamilton - Cayley identity.

(LA) Consider the algebra $Mat(N, A)$ of $n \times n$ matrices with coefficients in the algebra A. Fix a matrix $T \in Mat(n, A)$. Recall the notions of determinant $det(T)$. Define the characteristic polynomial $P_T \in A[\lambda]$, recall formulas for its coefficients.

We will consider the matrix $HC(T) := P_T(T) \in Mat(n, A)$.

1. Prove the **Theorem (Hamilton-Cayley identity).**

(HC) $\qquad HC(T) \equiv 0.$

Proof 1. You can argue along the following lines:

(i) Prove HC for diagonal matrices. Prove it for matrices conjugate to diagonal.

(ii) In case $A = \mathbf{C}$ show that matrices conjugate to diagonal are dense in he space of all matrices and deduce that HC holds for the algebra $\mathbf{C}$.

(iii) Let $\nu : A \to B$ be a morphism of algebras. Show that $HC(\nu(T)) = \nu(HC(T))$.

This implies that if ν is injective then HC for B implies HC for A. Similarly if ν is surjective than HC for A implies HC for B.

(iv) Show that it is enough to show HC for case when the algebra A is finitely generated over $\mathbf{Z}$.

(v) Show that if A is finitely generated over $\mathbf{Z}$ then we can find an algebra $D = \mathbf{Z}[x_i]$ that is mapped onto A and could be imbedded into $\mathbf{C}$. In particular HC for $\mathbf{C}$ implies HC for D implies HC for A.

[P] 2. Proof 2. Give another proof of HC along the following lines.

(i) Consider the algebra $Mat(n, A[\lambda]) = Mat(n, A)[\lambda]$. Using the ajugate matrix for the matrix $\lambda - T$ show that there exists a matrix $B = \sum B_i \lambda^i \in Mat(n, A)[\lambda]$ such that we have the following equality (*) in this algebra .

$$(*) \qquad (\lambda - T) \cdot B = P_T(\lambda) \cdot \mathbf{1}$$

(ii) Consider the centralizer C of the matrix T in $Mat(n, A)$. This is a subalgebra, not always commutative.

Show that all terms B_i in the equality (*) lie in $C[\lambda]$ so this is an equality in the algebra $C[\lambda]$.

(iii) Extend the natural morphism $C \to Mat(n, A)$ to a morphism of algebras $\nu : C[\lambda] \to Mat(n, A)$ by $\nu(\lambda) = T$. Show that ν maps both sides of the equality (*) to 0.

Remark. In fact the matrices B_i are uniquely defined by equation (*). Moreover, they lie in the commutative subalgebra $Z = A(T) \subset Mat(n, A)$ generated by T. Show these facts.

3. Let M be a finitely generated A-module and $T : M \to M$ its endomorphism.

(i) Show that there exists a monic polynomial $P \in A[t]$ such that the endomorphism $P(T)$ equals 0.

Hint. Using (HC) check this first for the case of a free module F. Then construct an endomorphism $U : F \to F$ of a free module and a surjective morphism $p : F \to M$ such that $Tp = pU$.

(ii) Suppose that we found an ideal $J \subset A$ such that $T(M) \subset JM$. Show that then we can choose the polynomial $P = t^n + \sum_1^n a_i t^{n-i}$ such that $P(T) = 0$ and the coefficients a_i lie in J.

[P] 4. Prove **Nakayama Lemma.** Let M be a finitely generated A-module and J an ideal of A such that $M = JM$. Then there exists an element $r \in 1 + J$ such that $rM = 0$.

Hint. You can use problem 3.

Another strategy is to use induction with respect to submodules of M. Namely, show that if L is a submodule of M, and for some element $r \in A$ we have $rM \subset L$ then $rL \subset rM = rJM = JrM \subset JL$.

[P] 5. Let M be a finitely generated A-module and T its endomorphism. Show that if T is surjective then it is invertible.

Show that analogues statement does not hold if we do not assume that M is finitely generated, or if we assume that T is injective instead of surjective.

Definition. A (commutative) A**-algebra** is a commutative ring B equipped with a homomorphism of rings $\nu : A \to B$. This naturally defines on B a structure of an A-module.

If $R \subset B$ is a subset in an A-algebra B we denote by $A < R >$ the subalgebra of B generated by image $\nu(A)$ and the subset R.

An A-algebra B is called **finitely generated** if $B = A < R >$ for some finite subset $R \subset B$.

An A-algebra B is called **finite** if B is finitely generated as an A-module (equivalent terminology - "B is finite over A" or "morphism $\nu : A \to B$ is finite").

6. (i) Consider morphisms of algebras $A \to B \to C$. Show that if C is finite over B and B is finite over A then C is finite over A.

(ii) Let B be an A-algebra, $x \in B$. Show that the following conditions are equivalent

(a) x is integral over A, i.e. there exists a monic polynomial $P \in A[t]$ such $P(x) = 0$.

(b) The algebra $C = A < x >$ is a finite over A.

(c) There exists a faithful C-module that is finitely generated as an A-module.

[P] (iii) Show that a finitely generated A-algebra B is finite over A iff all elements of B are integral over A.

Definition. A subset $S \subset A$ is called **multiplicatively closed** if $1 \in S$ and $s, t \in S$ implies that $st \in S$. A morphism $\nu : A \to B$ we will call S-inverting if $\nu(S) \subset B^*$.

7. (i) Show that the category of S-inverting morphisms has an initial object (i.e. an S-inverting morphism satisfying the universal property).

We will denote this morphism $i : A \to S^{-1}A$; the algebra $S^{-1}A$ is called the localization of A with respect to S.

(ii) Describe the kernel $K = Ker(i)$.

[P] 8. Let $\mathfrak{p} \subset A$ be an ideal.

(i) Show that $\mathfrak{p}$ is prime iff the complement $S_\mathfrak{p} = A \setminus \mathfrak{p}$ is multiplicatively closed.

In this case the algebra $S_\mathfrak{p}^{-1}A$ is called the **localization of A at the prime ideal $\mathfrak{p}$**. This is a very important notion, and for this algebra there is a standard notation $A_\mathfrak{p}$.

(ii) Show that $A_\mathfrak{p}$ is a local algebra, i.e. it has exactly one maximal ideal $\mathfrak{m}$. Show that $\mathfrak{m} = A_\mathfrak{p}\mathfrak{p}$. Describe its preimage $i^{-1}(\mathfrak{m})$ in A and the quotient field $K = A_\mathfrak{p}/\mathfrak{m}$.

[P] 9. Let $i : A \hookrightarrow B$ be a finite imbedding of algebras. Fix a morphism of algebras $\nu : A \to \Omega$, where Ω is an algebraically closed field.

(i) Show that there exists a non-zero morphism $v : B \to \Omega$ of A-modules.

Hint. Using localization reduce to the case when $K = \nu(A)$ is a subfield of Ω and then reduce to the case $A = K$.

(ii) Show that the space V of all morphisms v in (i) is a non-zero finite dimensional vector space over Ω. Show that the algebra B acts on this space. Show that eigenvectors of this action correspond to morphisms of algebras $\lambda : B \to \Omega$ that extend the morphism ν.

Deduce that the set of such morphisms $\lambda : B \to \Omega$ is finite and non-empty.

[P] 10. Let $i : A \hookrightarrow B$ be a finite imbedding of algebras.

Show that the map $\nu^{-1} : SpecB \to SpecA$ is onto and has finite fibers.

Show that the ideal $\mathfrak{p} \subset B$ is maximal iff the ideal $\nu^{-1}(\mathfrak{p}) \subset A$ is maximal.